JOURNAL OF DIFFERENTIAL EQUATIONS 9, 453–495 (1971)

Two-Point Problems for Abstract Evolution Equations*

JEFFERY M. COOPER

Department of Mathematics, University of Maryland, College Park, Maryland 20742

Received November 13, 1969

INTRODUCTION

The Cauchy problem, or initial value problem, for the operational differential equation

$$\frac{d}{dt} u(t) + A(t) u(t) = f(t)$$

has been studied by many authors. In this paper we shall adapt the variational method developed by Lions and others to solve a "two-point problem" for this equation.

Specifically, let H be a separable Hilbert space, with scalar product (f, g) and norm $|f|$, and let $[0, T]$ be a finite interval of the real line. For each $t \in [0, T]$, let $A(t)$ be an unbounded, self-adjoint operator in H with domain $D(A(t))$ which depends on t. We suppose that

(i) $A^{-1}(t)$ exists and is a bounded operator in H for all $t \in [0, T]$.

This implies that there is an orthogonal decomposition $H = H_+(t) \oplus H_-(t)$ which reduces $A(t)$, and such that

$$A_+(0) = A(0) \mid H_+(0) \cap D(A(0)) \text{ and } A_-(T) = - A(T) \mid H_-(T) \cap D(A(T))$$

are positive self-adjoint operators in $H_+(0)$ and $H_-(T)$, respectively. Let $P_+(0)\colon H \to H_+(0)$ and $P_-(T)\colon H \to H_-(T)$ be the orthogonal projections. We now pose the following problem:

($*$) Given $f(t) \in L^2(0, T; H)$, $u_0 \in D(A_+^{1/2}(0))$, and $u_T \in D(A_-^{1/2}(T))$, find a function $u \in L^2(0, T; D(A(t))) \cap H^1(0, T; H)$ such that

$$\frac{d}{dt} u(t) + A(t) u(t) = f(t)$$

* The author was supported by a National Science Foundation fellowship and a NATO postdoctoral fellowship while preparing the material for this paper, part of which was included in his doctoral thesis at the University of Illinois.

(in the sense of distributions on $(0, T)$ taking values in H) with

$$P_+(0)\,u(0) = u_0 \qquad \text{and} \qquad P_-(T)\,u(T) = u_T\,.$$

We shall prove the existence of a solution to this problem in Section 1, under the following hypotheses:

(ii) $t \to (A^{-1}(t)f, g) \in \mathscr{C}^1[0, 1]$ for each $f, g \in H$;

(iii) $|\, d/dt(A^{-1}(t)f, f)| \leqslant 2\alpha\, |f|_H^2,\ 0 \leqslant \alpha < 1$, for all $f \in H$.

For $u \in D(A(t))$, we may define the "absolute value" of $A(t)$ by

$$B(t)u = \int_{-\infty}^{\infty} |\lambda|\, dE_\lambda(t)u$$

where $\lambda \to E_\lambda(t)$ is the spectral resolution of $A(t)$. It follows that $B^{-1}(t)$ exists and is a bounded operator on H. In Section 2 we prove that the solution obtained in Section 1 is unique with the additional hypothesis

(iv) $t \to (B^{-1}(t)f, g) \in \mathscr{C}^1[0, T]$ for all $f, g \in H$.

In Section 3 we apply the methods and results of Sections 1 and 2 to systems of equations of the form

$$\frac{d}{dt}\,u(t) + A_1(t)\,u(t) + v(t) = f(t),$$

$$\frac{d}{dt}\,v(t) - A_2(t)\,v(t) + u(t) = g(t),$$

where $A_1(t)$ and $A_2(t)$ are families of positive self-adjoint operators satisfying (i), (ii), and (iii).

Section 4 is devoted to an adaptation of the Lions–Malgrange backwards uniqueness theorem to the solutions of problem $(*)$. We assume, in addition to (i)–(iv) that there is a constant $c > 0$ such that

(v) $|\, d/dt(A^{-1}(t)f, f)| \leqslant \epsilon\, |f|_H^2 + c\epsilon^{-1}\, |A^{-1}(t)f|_H^2$ for all $\epsilon, 0 < \epsilon \leqslant 1$.

We are then able to prove that if u is a solution of problem $(*)$ with $f(t) \equiv 0$, and $u(\tau) = 0$ for some $\tau \in [0, T]$, then $u(t) \equiv 0$ in $[0, T]$.

Section 5 discusses examples of operators which satisfy (i)–(iv) and in particular we prove a theorem which allows one to verify (iv) in many practical cases.

1. To prove the existence of solutions to problem $(*)$ we first state a fundamental theorem for functional equations, in the form due to Lions (see Lions, Ref. [14]).

Let F be a Hilbert space. For $u, v \in F$ the scalar product will be (u, v) and $|u|_F = (u, u)^{1/2}$ the norm. Let $\Phi \subset F$ be a linear subspace of F with scalar product $((\varphi, \psi))$ for $\varphi, \psi \in \Phi$, and norm $\|\varphi\| = ((\varphi, \varphi))^{1/2}$. Φ is not assumed to be complete. We do assume that the inclusion map of Φ into F is continuous. That is, there is a constant $c > 0$ such that

$$|\varphi|_F \leqslant c\|\varphi\| \qquad \text{for all } \varphi \in \Phi. \tag{1.1}$$

Now let $E(u, \varphi)$ be a sesquilinear form (linear in the first variable, conjugate linear in the second) defined on $F \times \Phi$. Suppose

(1.2) For each fixed $\varphi \in \Phi$, $u \to E(u, \varphi)$ is continuous on F;

(1.3) There is a constant $\alpha > 0$ such that for all $\varphi \in \Phi$,

$$|E(\varphi, \varphi)| \geqslant \alpha \|\varphi\|_\Phi^2 .$$

THEOREM 1.1. *Suppose the sesquilinear form E defined on $F \times \Phi$ satisfies* (1.1), (1.2), *and* (1.3). *Let L be any continuous conjugate linear form on Φ. Then there exists $u \in F$, possibly not unique, such that*

$$E(u, \varphi) = L(\varphi) \qquad \text{for all } \varphi \in \Phi$$

COROLLARY. *If u is the solution obtained by this theorem, then*

$$|u|_F \leqslant \frac{c}{\alpha}\|L\|,$$

where $\|L\| = \sup|L(\varphi)|$ *over* $\varphi \in \Phi$ *with* $\|\varphi\| = 1$.

Now let H be a separable Hilbert space with scalar product (f, g) and norm $|f| = (f, f)^{1/2}$, and let $[0, T]$ be a finite interval of the real line. Suppose that $t \to u(t) \in L^2(0, T; H)$ (for definition see Bourbaki, Ref. [4]). We say that $u' \in L^2(0, T; H)$ in the sense of distributions with values in H if there is a function $v(t) \in L^2(0, T; H)$ such that

$$-\int_0^T u(t)\, \varphi'(t)\, dt = \int_0^T v(t)\, \varphi(t)\, dt$$

for all $\varphi \in \mathscr{D}(0, T)$. This is equivalent to saying that

$$-\int_0^T (u(t), h)\, \varphi'(t)\, dt = \int_0^T (v(t), h)\, \varphi(t)\, dt$$

for all $h \in H$ and all $\varphi \in \mathscr{D}(0, T)$. We denote by $H^1(0, T; H)$ the vector space of all (classes) of functions $u \in L^2(0, T; H)$ such that $(d/dt)u \in L^2(0, T; H)$ in the distribution sense. $H^1(0, T; H)$ is a Hilbert space with the norm

$$\| u \|_{H^1} = \left(\int_0^T | u |_H^2 \, dt + \int_0^T | u' |^2 \, dt\right)^{1/2}.$$

It is well known that if $u \in H^1(0, T; H)$, then u is equal a.e. to a continuous function with values in H. Thus the point value $u(0)$ is well-defined, and it can be shown that the function $u \to u(0)$ is continuous from $H^1(0, T; H)$ onto H.

Now for each $t \in [0, T]$, let $A(t)$ be a closed self-adjoint operator with domain $D(A(t))$ dense in H. Assume

(1.4) For each $t \in [0, T]$, $A^{-1}(t)$ exists and is a bounded operator on H;

(1.5) For each pair $f, g \in H$, the function $t \to (A^{-1}(t)f, g)$ is continuously differentiable on $[0, T]$.

The following lemma is a simple consequence of (1.4), (1.5), and the principle of uniform boundedness (see Riesz and Sz.-Nagy, Ref. [18]).

LEMMA 1.1. (i) *$d/dt(A^{-1}(t)f, g) = (\dot{A}^{-1}(t)f, g)$, where $\dot{A}^{-1}(t)$ is a family of bounded symmetric operators in H.*

(ii) *There is a constant $c > 0$ such that the operator norm $\| A^{-1}(t)\| \leqslant c$ for all $t \in [0, T]$.*

(iii) *$t \to A^{-1}(t)$ is continuous in the uniform operator norm of $\mathscr{L}(H, H)$, the space of bounded linear operators on H.*

Now part (iii) of Lemma 1.1 implies that there is a $\delta > 0$, independent of t, such that for $u \in D(A(t))$, $| A(t)u | \geqslant \delta | u |_H$. Thus $((u, v)) = (A(t)u, A(t)v)$ defines a scalar product on $D(A(t))$ which yields a norm equivalent to the graph norm. We define $L^2(0, T; D(A(t)))$ as the space of functions $u(t) \in L^2(0, T; H)$ such that $u(t) \in D(A(t))$ a.e. and $A(t)\, u(t) \in L^2(0, T; H)$. $A(t)$ is closed for each $t \in [0, T]$, and hence by the remark following Lemma 1.1, $L^2(0, T; D(A(t)))$ is a Hilbert space with the norm

$$\| u \| = \left(\int_0^T | A(t)\, u(t)|^2 \, dt\right)^{1/2}.$$

Finally, we set $W = L^2(0, T; D(A(t))) \cap H^1(0, T; H)$. W is a Hilbert space with the norm

$$\| u \|_W = \left(\int_0^T | A(t)\, u(t)|^2 \, dt + \int_0^T | u'(t)|^2 \, dt\right)^{1/2}.$$

Let $\lambda \to E_\lambda(t)$ be the spectral resolution of $A(t)$. Since $| A(t)u | \geqslant \delta | u |$, it follows that for each t, $\lambda \to E_\lambda(t)$ is constant on the interval $(-\delta, \delta)$, i.e., for $u \in D(A(t))$,

$$A(t)u = \int_{-\infty}^{-\delta} \lambda dE_\lambda(t)u + \int_{\delta}^{\infty} \lambda dE_\lambda(t)u.$$

Let $H_-(t)$ be the range of the orthogonal projection $P_-(t) = E_0(t)$ and $H_+(t)$ its orthogonal complement with projection $P_+(t) = I - P_-(t)$. We allow the possibility that for some $t \in [0, T]$, either $H_+(t) = \{0\}$ or $H_-(t) = \{0\}$. Then for $u \in D(A(t))$,

$$\begin{aligned}(A(t)\, P_-(t)u, u) &= \int_{-\infty}^{-\delta} \lambda d(E_\lambda(t)\, E_0(t)u, u) + \int_{\delta}^{\infty} \lambda d(E_\lambda(t)\, E_0(t)u, u) \\ &= \int_{-\infty}^{-\delta} d(E_\lambda(t)u, u) \leqslant -\delta\, | P_-(t)u |^2,\end{aligned}$$

and similarly $(A(t)\, P_+(t)u, u) \geqslant \delta \mid P_+(t)u \mid^2$. Thus, we may write

$$A(t)u = A_+(t)\, P_+(t)u - A_-(t)\, P_-(t)u, \qquad u \in D(A(t)),$$

where $A_+(t)$ and $A_-(t)$ are positive self-adjoint operators in $H_+(t)$ and $H_-(t)$, respectively, with domains

$$D(A_+(t)) = H_+(t) \cap D(A(t)) \qquad \text{and} \qquad D(A_-(t)) = H_-(t) \cap D(A(t)).$$

Note that because $A_+(t)$ is a positive self-adjoint operator in $H_+(t)$, we may take the square root. $A_+^{1/2}(t)$ is again a positive self-adjoint operator in $H_+(t)$ satisfying $| A_+^{1/2}(t)u |^2 = (A_+(t)u, u) \geqslant \delta \mid u \mid^2$ for $u \in D(A_+(t))$ and similarly for $A_-(t)$ (see Riesz and Sz.-Nagy, Ref. [18]).

To obtain the existence of solutions to problem $(*)$ we assume

(1.6) For each $f \in H$, $| d/dt(A^{-1}(t)f, f)| \leqslant 2\alpha\, | f |^2$, $0 \leqslant \alpha < 1$, independent of $t \in [0, T]$.

The proof of the following theorem is adopted from Lions (see Lions, Ref. [14, Chap. VII]).

THEOREM 1.2. *Assume that* $A(t)$ *satisfies* (1.4), (1.5), (1.6). *Let* $f \in L^2(0, T; H)$ *be given along with* $u_0 \in D(A_+^{1/2}(0))$ *and* $u_T \in D(A_-^{1/2}(T))$. *Then there is a function* $u \in W$ *such that*

(i) $Au + u' = f$ *in* $\mathscr{D}'(0, T; H)$;

(ii) $P_+(0)\, u(0) = u_0$;

(iii) $P_-(T)\, u(T) = u_T$.

Proof. We shall use Theorem 1.1. Let $F = L^2(0, T; D(A(t)))$, and let Φ be the subspace of functions $\varphi \in F$ such that $d/dt(A\varphi) \in L^2(0, T; H)$, $\varphi(0) \in H_+(0)$, and $\varphi(T) \in H_-(T)$. We give Φ the norm

$$\| \varphi \|_\Phi = (\| \varphi \|_F^2 + | A_+^{1/2}(0)\, \varphi(0)|_H^2 + | A_-^{1/2}(T)\, \varphi(T)|_H^2)^{1/2},$$

which gives Φ a topology finer than that induced from F. On $F \times \Phi$ we define

$$E(u, \varphi) = \int_0^T (Au, A\varphi) - (u, D(A\varphi))\, dt, \qquad \text{where} \quad D = \frac{d}{dt}.$$

$u \to E(u, \varphi)$ is continuous on F for each fixed $\varphi \in \Phi$. To show that E is coercive on Φ, i.e., verifies (1.3), we examine

$$2 \operatorname{Re} \int_0^T (\varphi, D(A\varphi))\, dt = 2 \operatorname{Re} \int_0^T (A^{-1}\psi, \psi')\, dt, \qquad \psi = A\varphi.$$

Now

$$\begin{aligned}
&- 2 \operatorname{Re} \int_0^T (A^{-1}\psi, \psi')\, dt \\
&\quad = - \int_0^T \{(A^{-1}\psi, \psi') + (\psi', A^{-1}\psi)\}\, dt \\
&\quad = -(A^{-1}\psi, \psi)|_0^T + \int_0^T (D(A^{-1}\psi), \psi)\, dt - \int_0^T (\psi', A^{-1}\psi)\, dt \\
&\quad = (A_+(0)\, \varphi(0), \varphi(0)) + (A_-(T)\, \varphi(T), \varphi(T)) + \int_0^T (\dot{A}^{-1}\psi, \psi)\, dt \\
&\quad \geqslant | A_+^{1/2}(0)\, \varphi(0)|^2 + | A_-^{1/2}(T)\, \varphi(T)|^2 - 2\alpha \int_0^T | A\varphi |^2\, dt,
\end{aligned}$$

where the last inequality follows from (1.6). Thus

$$\operatorname{Re} E(\varphi, \varphi) \geqslant (1 - \alpha) \int_0^T | A\varphi |^2\, dt + \tfrac{1}{2}\{| A_+^{1/2}(0)\, \varphi(0)|^2 + | A_-^{1/2}(T)\, \varphi(T)|^2\},$$

which proves E coercive on Φ. Now take

$$L(\varphi) = \int_0^T (f, A\varphi)\, dt + (u_0, A_+(0)\, \varphi(0)) + (u_T, A_-(T)\, \varphi(T)),$$

which is a continuous linear form on Φ. Then by Theorem 1.1 there is a function $u \in F$ such that $E(u, \varphi) = L(\varphi)$ for all $\varphi \in \Phi$. Therefore

$$(1.7) \quad \int_0^T (Au, \psi) - (u, \psi')\, dt = \int_0^T (f, \psi)\, dt + (u_0, \psi(0)) - (u_T, \psi(T))$$

for all $\psi = A\varphi$, $\varphi \in \Phi$. That is, for all functions in Ψ, the space of $\psi \in L^2(0, T; H)$ such that $\psi' \in L^2(0, T; H)$, with $\psi(0) \in H_+(0)$ and $\psi(T) \in H_-(T)$. In particular Ψ contains all functions of the form

$$\psi = \theta \otimes v, \qquad \text{where } v \in H \text{ and } \theta \in \mathscr{D}(0, T).$$

From (1.7) we deduce that

$$\int_0^T \{(A(t)\, u(t), v)\, \bar{\theta}(t) - (u(t), v)\, \bar{\theta}'(t)\}\, dt = \int_0^T (f(t), v)\, \bar{\theta}(t)\, dt$$

for all $v \in H$ and $\theta \in \mathscr{D}(0, T)$. This implies that $Au + u' = f$ in $\mathscr{D}'(0, T; H)$, whence $u' \in L^2(0, T; H)$. Thus $u(0)$ and $u(T)$ are well-defined elements in H. Integrating by parts and substituting in (1.7) we obtain

$$(u(0), \psi(0)) - (u(T), \psi(T)) = (u_0\,, \psi(0)) - (u_T\,, \psi(T)),$$

holding for all $\psi \in \Psi$. This implies that

$$(u(0) - u_0\,, v) = 0$$

and

$$(u(T) - u_T\,, w) = 0 \qquad \text{for all } v \in H_+(0) \text{ and for all } w \in H_-(T),$$

whence

$$P_+(0)\, u(0) = u_0 \qquad \text{and} \qquad P_-(T)\, u(T) = u_T\,.$$

Thus, Theorem 1.2 is proved.

2. Our next goal is to show that the solution to problem (*) found in Theorem 1.2 is unique. In the case when $A(t)$ is uniformly semibounded from below and the solution depends only on initial Cauchy data, uniqueness is easily obtained, using the usual "energy integral" method. This method relies on the fact that for some $k \geqslant 0$, $A(t) + k \geqslant 0$ for all $t \in [0, T]$. For our problem the situation is a bit more involved and we need first to prove a result about the trace of a function in W.

We begin by defining an operator related to $A(t)$. For $u \in D(A(t))$, set

$$B(t)u = \int_{-\infty}^{\infty} |\, \lambda \,|\, dE_\lambda(t)u = A_-(t)\, P_-(t)u + A_+(t)\, P + (t)u.$$

$B(t)$ is clearly a self-adjoint operator in H with domain $D(A(t))$. $(B(t)u, u) \geqslant \delta(|\, P_+(t)u\,|^2 + |\, P_-(t)u\,|^2) = \delta\, |\, u\,|^2$ for $u \in D(A(t))$ so that $B(t)^{-1}$ exists and is a bounded symmetric operator in H. Furthermore, the graph norm on $D(B^{1/2}(t))$ is equivalent to $|\, B^{1/2}(t)u\,|_H$, $u \in D(B^{1/2}(t))$.

We shall assume

(2.1) For each $f, g \in H$, $t \to (B^{-1}(t)f, g)$ is continuously differentiable on $[0, T]$.

It follows that $t \to B^{-1}(t)$ is continuous in the operator norm of $\mathscr{L}(H, H)$ on $[0, T]$. We also note that for $u \in D(A(t)) = D(B(t))$, we have $| B(t)u |^2 = | A(t)u |^2$. Hence

$$L^2(0, T; D(B(t))) = L^2(0, T; D(A(t)))$$

with the same norm. Finally we also note that the space W is the same for both $A(t)$ and $B(t)$. We refer the reader to Lions, Ref. [14, Chapter IV] for a proof of the following version of Friedrich's Lemma.

LEMMA 2.1. *Let $R(t)$, $-\infty < t < \infty$, be a family of bounded linear operators in a Hilbert space H such that $t \to (R(t)f, g)$ is continuous on $(-\infty, \infty)$ for each pair $f, g \in H$. Suppose also that the distribution derivative of $(R(t)f, g)$ is measurable and*

$$\left| \frac{d}{dt}(R(t)f, g) \right| = |(\dot{R}(t)f, g)| \leqslant C \, | f |_H \, | g |_H .$$

Let $\rho(t)$ be a real-valued $\mathscr{C}^\infty$ function with support in $[-1, +1]$ with $\rho(t) \geqslant 0$ and $\rho(t) = \rho(-t)$. Finally, suppose $\int \rho \, dt = 1$ and set $\rho_m(t) = m\rho(mt)$. Then for $u \in L^2(-\infty, \infty; H)$ one has

$$\frac{d}{dt}[R(t)(u * \rho_m) - (R(t)u) * \rho_m] \to 0 \qquad \text{as} \qquad m \to \infty$$

in $L^2(-\infty, \infty; H)$.

Now set $W_0 = \{u \in W : (Au)' \in L^2(0, T; H)\}$. Using Lemma 2.1 we shall prove

LEMMA 2.2. *Assume that $A(t)$ satisfies* (1.4) *and* (1.5). *Then W_0 is dense in W.*

Proof. We extend $A(t)$ to all of $\mathbf{R} = (-\infty, +\infty)$ as follows:

$$\tilde{A}(t) = \begin{cases} A(T) & \text{for } \; t \leqslant -T, \\ A(-t) & \text{for } \; -T \leqslant t \leqslant 0, \\ A(t) & \text{for } \; 0 \leqslant t \leqslant T, \\ A(2T - t) & \text{for } \; T \leqslant t \leqslant 2T, \\ A(0) & \text{for } \; t \geqslant 2t, \end{cases}$$

with $D(\tilde{A}(t))$ defined accordingly. Extended in this way $t \to (\tilde{A}^{-1}(t)f, g)$ is continuous and piecewise continuously differentiable on $\mathbf{R}$ for any $f, g \in H$. Next, given $u \in W$, we extend u first by reflection to $[-T, 2T]$. That is,

$$u^*(t) = \begin{cases} u(-t) & \text{for } -T \leqslant t \leqslant 0, \\ u(t) & \text{for } 0 \leqslant t \leqslant T, \\ u(2T - t) & \text{for } T \leqslant t \leqslant 2T. \end{cases}$$

Now let $\varphi(t)$ be a real-valued $\mathscr{C}^\infty$ function such that $\varphi = 1$ on $[0, T]$ and $\varphi(t) = 0$ for $t \leqslant -\frac{1}{2}T$ and $t \geqslant \frac{3}{2}T$. We set $\tilde{u}(t) = \varphi(t)\, u^*(t)$. Then $\tilde{u}' \in L^2(-\infty, +\infty; H)$ and $\tilde{u}(t) \in D(\tilde{A}(t))$ a.e. with $\tilde{A}\tilde{u} \in L^2(-\infty, \infty; H)$. We shall approximate $\tilde{A}\tilde{u}$ by a smooth function on all of $(-\infty, +\infty)$. With ρ_n as in Lemma 2.1, set $v_n = \tilde{A}^{-1}((\tilde{A}\tilde{u})^* \rho_n)$. Clearly $v_n(t) \in D(\tilde{A}(t))$ a.e. and $\tilde{A}v_n = (\tilde{A}\tilde{u})^* \rho_n \in L^2(-\infty, +\infty; H)$.
Furthermore,

$$v_n' = (\tilde{A}^{-1})^{\cdot}((\tilde{A}\tilde{u})^* \rho_n) + \tilde{A}^{-1}((\tilde{A}\tilde{u})^* \rho_n') \in L^2(-\infty, \infty; H)$$

because

$$\left| \frac{d}{dt} (A^{-1}(t)f, g) \right| \leqslant C \,|\, f \,|_H \,|\, g \,|_H \qquad \text{for} \quad -\infty < t < \infty$$

by Lemma 1.1. Now $\tilde{A}v_n = (\tilde{A}\tilde{u})^* \rho_n \to (\tilde{A}\tilde{u})$ in $L^2(-\infty, \infty; H)$ as $n \to \infty$. It remains to be shown that $v_n' \to \tilde{u}'$. We have

$$\begin{aligned} v_n' &= \frac{d}{dt} (\tilde{A}^{-1}((\tilde{A}\tilde{u})^* \rho_n)) \\ &= \frac{d}{dt} (\tilde{u}^* \rho_n) + \frac{d}{dt} [\tilde{A}^{-1}((\tilde{A}\tilde{u})^* \rho_n) - (\tilde{A}^{-1}(\tilde{A}\tilde{u}))^* \rho_n]. \end{aligned}$$

The first term converges in $L^2(-\infty, \infty; H)$ to $\tilde{u}'$ because $\tilde{u}' \in L^2(-\infty, +\infty; H)$, and the second term converges to zero by Lemma 2.1. The restriction of v_n to $[0, T]$ belongs to W_0 and converges in the norm of W to $\tilde{u}\,|_{[0,T]} = u$.

For completeness, we prove the following trace theorem for W, due to Lions (see Ref. [14, Chap. VII]).

Lemma 2.3. *Assume that $A(t)$ satisfies* (1.4), (1.5), *and* (2.1). *Then the map $u \to u(0)$ is continuous from W into $D(B^{1/2}(0))$ with the Hilbert graph norm $\| v \| = | B^{1/2}(0)v |$.*

Proof. Consider first $u \in W_0$. Let φ be a $\mathscr{C}^\infty$ function, $0 \leqslant \varphi \leqslant 1$, such that $\varphi(0) = 1$ and $\varphi(T) = 0$. Set $v = \varphi u$. Then $v \in W_0$ also and the following calculation is valid:

$$2 \operatorname{Re} \int_0^T (Bv, v')\,dt = \int_0^T (w, (B^{-1}w)')\,dt + \int_0^T ((B^{-1}w)', w)\,dt$$
$$= (w, B^{-1}w)|_0^T + \int_0^T (\dot{B}^{-1}w, w)\,dt, \qquad w = Bv.$$

That is,

$$\begin{aligned} &| B^{1/2}(0)\, u(0)|^2 \\ &\quad = (B(0)\, v(0), v(0)) \\ &\quad = \int_0^T \varphi^2(\dot{B}^{-1}Bu, Bu)\,dt - 2 \operatorname{Re} \int_0^T \varphi^2(Bu, u')\,dt - 2 \int_0^T \varphi\varphi'(Bu, u)\,dt \\ &\quad \leqslant (1 + c + c_1) \int_0^T | Bu |^2\,dt + \int_0^T | u' |^2\,dt + c_1 \int_0^T | u |^2\,dt \end{aligned}$$

where $c = \sup_{0 \leqslant t \leqslant T} | \dot{B}^{-1}(t)|$ and $c_1 = \sup_{0 \leqslant t \leqslant T} | \varphi\varphi' |$. Hence there is a constant $c_2 > 0$ such that

$$| B^{1/2}(0)\, u(0)|^2 \leqslant c_2 \| u \|^2_{W(0,T)}$$

and $u \to u(0)$ is continuous from W_0 into $D(B^{1/2}(0))$. It follows from the density of W_0 in W and the completeness of $D(B^{1/2}(0))$ that this map has a unique extension to all of W.

Now we observe that $D(B^{1/2}(t))$ with the Hilbert graph norm isomorphic to $D(A_+^{1/2}(t)) \times D(A_-^{1/2}(t))$ with the product topology. In fact, if $u \in D(B(t))$, then

$$| B^{1/2}(t)u |^2 = (B(t)u, u) = (A_+(t)\, P_+(t)u, u) + (A_-(t)\, P_-(t)u, u)$$
$$= | A_+^{1/2}(t)\, P_+(t)u |^2 + | A_-^{1/2}(t)\, P_-(t)u |^2.$$

The equality extends, on the left, to $D(B^{1/2}(t))$ and on the right, to $D(A_+^{1/2}(t)) \times D(A_-^{1/2}(t))$ because $D(B(t))$ is dense in $D(B^{1/2}(t))$, and $D(A_+(t))$ (resp. $D(A_-(t))$) is dense in $D(A_+^{1/2}(t))$ (resp. $D_-(A_-^{1/2}(t))$). (See Ref. [18]).

Now for $u \in D(B^{1/2}(t)) = D(A_+^{1/2}(t)) \times D(A_-^{1/2}(t))$ let us set

$$Q(t, u) = | A_+^{1/2}(t)\, P_+(t)u |^2 - | A_-^{1/2}(t)\, P_-(t)u |^2.$$

For $u \in D(A(t))$, $Q(t, u) = (A(t)u, u)$. Clearly $Q(t, u)$ is a continuous quadratic form on $D(B^{1/2}(t))$.

The following Corollary is a consequence of Lemma 2.3.

COROLLARY. *Suppose $A(t)$ satisfies* (1.4), (1.5), *and* (2.1). *Then for $u \in W$ we have (with $v = Au$)*

$$2\,\mathrm{Re}\int_0^T (Au, u')\,dt = \int_0^T (\dot{A}^{-1}v, v)\,dt + Q(T, u(T)) - Q(0, u(0)). \tag{2.2}$$

Proof. For $u \in W$, there is a sequence $u_n \in W_0$ such that $u_n \to u$ in W, by Lemma 2.2. For each u_n we may write

$$\begin{aligned} 2\,\mathrm{Re}\int_0^T (Au_n, u_n')\,dt &= \int_0^T (\dot{A}^{-1}v_n, v_n)\,dt \\ &\quad + (A(T)\,u_n(T), u_n(T)) - (A(0)\,u_n(0), u_n(0)) \\ &= \int_0^T (\dot{A}^{-1}v_n, v_n) + Q(T, u_n(T)) - Q(0, u_n(0)) \end{aligned} \tag{2.3}$$

where $v_n = Au_n$. Passing to the limit in W, and using Lemma 2.3, as well as the continuity of Q on $D(B^{1/2}(t))$, we obtain (2.2).

Uniqueness of the solution in Theorem 1.2 will follow from

THEOREM 2.1. *Suppose that $A(t)$ satisfies* (1.4), (1.5), (1.6), *and* (2.1). *Let $u \in W$ be a nontrivial solution of $Au + u' = 0$ on $(0, T)$. Then $Q(T, u(T)) < Q(0, u(0))$.*

Proof. Suppose $u \in W$ satisfies $Au + u' = 0$. Multiplying by Au and integrating we have

$$\begin{aligned} 0 &= 2\int_0^T |Au|^2\,dt + 2\,\mathrm{Re}\int_0^T (u', Au)\,dt \\ &= 2\int_0^T |Au|^2\,dt + \int_0^T (\dot{A}^{-1}Au, Au)\,dt + Q(T, u(T)) - Q(0, u(0)) \\ &\geqslant 2(1-\alpha)\int_0^T |Au|^2\,dt + Q(T, u(T)) - Q(0, u(0)), \end{aligned}$$

where we have used the corollary to Lemma 2.3 and (2.2). Clearly, then, if $Q(T, u(T)) \geqslant Q(0, u(0))$ we have $\int_0^T |Au|^2\,dt \leqslant 0$, whence $u = 0$.

THEOREM 2.2. *Suppose that $A(t)$ verifies* (1.4), (1.5), (1.6), *and* (2.1). *Then the solution obtained in Theorem* 1.2 *is unique and depends continuously on the data in the sense that the map $(u_0, u_T, f) \to u$ is continuous from $D(A_+^{1/2}(0)) \times D(A_-^{1/2}(T)) \times L^2(0, T; H)$ into $L^2(0, T; D(A(t)))$.*

Proof. Suppose $u_0 \in D(A_+^{1/2}(0))$, $u_T \in D(A_-^{1/2}(T))$, and $f \in L^2(0, T; H)$ are given. Suppose $u, v \in W$ are two solutions. Then $w = u - v$ satisfies $Aw + w' = 0$, $P_+(0)\, w(0) = P_-(T)\, w(T) = 0$. But then

$$Q(T, w(T)) - Q(0, w(0)) = | A_+^{1/2}(T)\, w(T)|^2 + | A_-^{1/2}(0)\, w(0)|^2 \geqslant 0,$$

whence $w = 0$ by Theorem 2.1. The continuous dependence of the solution follows immediately from the corollary to Theorem 1.1 and the definition of the linear form L in the proof of Theorem 1.2.

We emphasize that $A(t)$ is not required to be semibounded from below in Theorems 1.2, 2.1, and 2.2, and indeed if $A(t)$ is semibounded for each t, it need not be so uniformly on $[0, T]$. Suppose we assume, in addition to (1.4), (1.5), (1.6), and (2.1), that

(2.4) $H_-(t) \neq \{0\}$ for $0 \leqslant t < T$ and $H_-(T) = \{0\}$, i.e., $H_+(T) = H$.

This of course implies that $A(T) = A_+(T) > 0$ and $B(T) = A(T)$. An immediate consequence of (2.4) is

(2.5) $\lim_{t \to T} A_-^{-1}(t)\, P_-(t) = 0$ in the operator norm of $\mathscr{L}(H, H)$.

In fact,

$$2 \lim_{t \to T} A_-^{-1}(t)\, P_-(t) = \lim_{t \to T} B^{-1}(t) - A^{-1}(t) = 0$$

because $t \to A^{-1}(t)$ and $t \to B^{-1}(t)$ are both continuous in the operator norm of $\mathscr{L}(H, H)$. It follows from (2.5) that the family $A(t)$ cannot be uniformly semibounded from below on $[0, T]$. For $u \in D(A_-(t))$, $0 \leqslant t \leqslant T$, we have

$$(A(t)u, u) = -(A_-(t)u, u) \leqslant - \| A_-^{-1}(t)\|^{-1}\, | u |^2,$$

and by (2.5), $\| A_-^{-1}(t)\|^{-1} \to \infty$ as $t \to T$.

Combining Theorems 1.2 and 2.2 for this case, we obtain

THEOREM 2.3. *Suppose $A(t)$ satisfies* (1.4), (1.5), (1.6), (2.1), *and* (2.4). *Suppose $u_0 \in D(A_+^{1/2}(0))$ and $f \in L^2(0, T; H)$ given. Then there is one and only one function $u \in W$ such that*

(i) $Au + u' = f$;

(ii) $P_+(0)\, u(0) = u_0$,

and the correspondence $(u_0, f) \to u$ is continuous from $D(A_+^{1/2}(0)) \times L^2(0, T; H)$ into W.

This is the result announced in Ref. [8] for the Cauchy problem when the family of operators $A(t)$ is not uniformly semibounded from below.

We give an example of this behavior in Section 5.

Using the method of proof of Theorems 1.2 and 2.2, it is possible to find unique solutions to a problem of "periodic boundary conditions."

THEOREM 2.4. *Suppose that* $A(t)$ *satisfies* (1.4), (1.5), (1.6), (2.1), *and* $A(0) = A(T)$. *Then for* $f \in L^2(0, T; H)$ *given, there is a unique function* $u \in W$ *such that*

(i) $Au + u' = f$;

(ii) $u(0) = u(T)$,

and $\| u \|_W \leqslant c \| f \|_{L^2(0,T;H)}$.

3. In this section we consider systems of evolution equations which may be studied in the framework of two-point problems. With H a separable Hilbert space as before, and $[0, T]$ a finite interval of the real line, let $A_1(t)$ and $A_2(t)$ be two families of self-adjoint operators in H with domains $D(A_1(t))$ and $D(A_2(t))$. Suppose that

(3.1) $(A_i(t)u, u) \geqslant c_i \mid u \mid^2$ for all $u \in D(A_i(t))$ $(i = 1, 2)$, where $c_i > 0$ is a constant independent of t.

It follows from (3.1) that $A_1(t)$ and $A_2(t)$ have bounded inverses in H for each $t \in [0, T]$. We then assume (for $i = 1, 2$)

$$\frac{1}{c_1} + \frac{1}{c_2} = 2\gamma \qquad \text{where} \quad \gamma < 1; \tag{3.2}$$

$$t \to (A_i^{-1}(t) f, g) \in \mathscr{C}^1[0, T] \qquad \text{for} \quad f, g \in H. \tag{3.3}$$

Consider now the system of evolution equations

$$\begin{cases} \dfrac{d}{dt} u(t) + A_1(t)\, u(t) + v(t) = f(t), \\ \dfrac{d}{dt} v(t) - A_2(t)\, v(t) + u(t) = g(t), \end{cases} \tag{3.4}$$

which we shall now rewrite as a vector equation. Let $\mathscr{H} = H \oplus H$. A vector $\mathscr{u} \in \mathscr{H}$ with components u_1 and u_2 will be written $\mathscr{u} = [u_1, u_2]$. The scalar product in $\mathscr{H}$ will be written

$$((\mathscr{u}, \mathscr{v})) = (u_1, v_1) + (u_2, v_2) \qquad \text{for} \quad \mathscr{u} = [u_1, u_2],\ \mathscr{v} = [v_1, v_2],$$

$$\text{with} \quad \| \mathscr{u} \| = ((\mathscr{u}, \mathscr{u}))^{1/2}.$$

With this scalar product, $\mathscr{H}$ is a Hilbert space. Let $\mathscr{A}(t)$ be the matrix of operators

$$\mathscr{A}(t) = \begin{bmatrix} A_1(t) & I \\ I & -A_2(t) \end{bmatrix}, I = \text{identity in } H,$$

with domain $D(\mathscr{A}(t)) = D(A_1(t)) \oplus D(A_2(t))$. Then the system (3.4) may be written as the evolution equation

$$\frac{d}{dt}\,\mathscr{u}(t) + \mathscr{A}(t)\,\mathscr{u}(t) = \mathscr{f}(t), \tag{3.5}$$

where $\mathscr{u}(t) = [u(t), v(t)]$ and $\mathscr{f}(t) = [f(t), g(t)]$.

LEMMA 3.1. *Assuming* (3.1), *the operator $\mathscr{A}(t)$ with domain $D(\mathscr{A}(t)) = D(A_1(t)) \oplus D(A_2(t))$ is self-adjoint and has a continuous inverse in $\mathscr{H}$. If in addition* (3.2) *is assumed, then*

$$\| \mathscr{A}(t)\mathscr{u} \|^2 \geqslant (1-\gamma)^2 (| A_1(t)u |^2 + | A_2(t)v|^2) \qquad \textit{for} \quad \mathscr{u} = [u, v] \in D(\mathscr{A}(t)).$$

Proof. $\mathscr{A}(t)$ is clearly self-adjoint. Let E be the operator in $\mathscr{H}$ given by

$$E = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, I = \text{identity in } H.$$

E is an isometry of $\mathscr{H}$ onto $\mathscr{H}$. Now the operator $E\mathscr{A}(t)$ is one-to-one on $D(\mathscr{A}(t))$ because

$$\begin{aligned}\operatorname{Re}((E\mathscr{A}(t)\mathscr{u}, \mathscr{u})) &= (A_1(t)u, u) + (A_2(t)v, v) \\ &\geqslant c_1 \,| u |^2 + c_2 \,| v |^2, \qquad \mathscr{u} = [u, v] \in D((t)).\end{aligned}$$

Thus, $\mathscr{A}(t)$ is one-to-one, and therefore invertible in $\mathscr{H}$.

To prove the second statement of the lemma, let

$$\mathscr{C}(t) = \mathscr{A}(t) \begin{bmatrix} A_1^{-1}(t) & 0 \\ 0 & -A_2^{-1}(t) \end{bmatrix}.$$

$\mathscr{C}(t)$ is clearly bounded. Furthermore, for $\mathscr{u} = [u, v] \in \mathscr{H}$,

$$\begin{aligned}\operatorname{Re}((\mathscr{C}(t)\,\mathscr{u}, \mathscr{u})) &= | u |^2 + | v |^2 + \operatorname{Re}(A_1^{-1}(t)u, v) - \operatorname{Re}(A_2^{-1}(t)v, u) \\ &\geqslant (1-\gamma)(| u |^2 + | v |^2) = (1-\gamma)\, \| \mathscr{u} \|^2\end{aligned}$$

by (3.2). Hence $\mathscr{C}(t)$ is maximal accretive and the spectrum of $\mathscr{C}(t)$ is contained in the numerical range. Therefore $\| \mathscr{C}(t)\mathscr{u} \| \geqslant (1-\gamma) \| \mathscr{u} \|$ for all $\mathscr{u} \in \mathscr{H}$. Now for $\mathscr{u} = [u, v] \in D(\mathscr{A}(t))$ and $\mathscr{v} = [A_1(t)u, A_2(t)v]$,

$$\| \mathscr{A}(t)\mathscr{u} \|^2 = \| \mathscr{C}(t)\mathscr{v} \|^2 \geqslant (1-\gamma) \| \mathscr{v} \|^2 \geqslant (1-\gamma)^2 (| A_1(t)u |^2 + | A_2(t)v |^2),$$

which proves the lemma.

It follows that

$$\mathscr{W} = L^2(0, T; D(\mathscr{A}(t))) \cap H^1(0, T; \mathscr{H}) = W_1 \oplus W_2 ,$$

where

$$W_1 = L^2(0, T; D(A_1(t))) \cap H^1(0, T; H),$$
$$W_2 = L^2(0, T; D(A_2(t))) \cap H^1(0, T; H).$$

In order to obtain theorems of existence and uniqueness for the system (3.4) (or equivalently the equation (3.5)), we must make the following additional hypothesis:

$$(3.6) \qquad \left| \frac{d}{dt} (A_1^{-1}(t)f, f) \right| + \left| \frac{d}{dt} (A_2^{-1}(t)g, g) \right| \leqslant 2\alpha(1-\gamma)(|f|^2 + |g|^2), \qquad f, g \in H,$$

where $\alpha < 1$, independent of t.

Let $\mathscr{H}_+(0)$ and $\mathscr{H}_-(T)$ be the orthogonal subspaces defined in Section 1 such that

$$\mathscr{A}_+(0) = \mathscr{A}(0) \mid \mathscr{H}_+(0) \cap D(\mathscr{A}(0))$$

and

$$\mathscr{A}_-(T) = -\mathscr{A}(T) \mid \mathscr{H}_-(T) \cap D(\mathscr{A}(T))$$

are positive self-adjoint operators in $\mathscr{H}_+(0)$ and $\mathscr{H}_-(T)$, respectively. Let $\mathscr{P}_+(0)$ and $\mathscr{P}_-(T)$ be the corresponding orthogonal projections.

THEOREM 3.1. *Suppose that $A_1(t)$ and $A_2(t)$ are two families of self-adjoint operators in H which satisfy* (3.1), (3.2), (3.3), *and* (3.6). *Let $\mathscr{u}_0 \in D(\mathscr{A}_+^{1/2}(0))$ and $\mathscr{u}_T \in D(\mathscr{A}_-^{1/2}(T))$ be given, as well as $\mathscr{f}(t) = [f(t), g(t)] \in L^2(0, T; \mathscr{H})$. Then there is one and only one solution $\mathscr{u} \in \mathscr{W} = W_1 \oplus W_2$ of the system* (3.4) *with $\mathscr{P}_+(0)\, \mathscr{u}(0) = \mathscr{u}_0$ and $\mathscr{P}_-(T)\, \mathscr{u}(T) = \mathscr{u}_T$. The solution depends continuously on the given data in that $(\mathscr{u}_0, \mathscr{u}_T, \mathscr{f}) \to \mathscr{u}$ is continuous from*

$$D(\mathscr{A}_+^{1/2}(0)) \times D(\mathscr{A}_-^{1/2}(T)) \times L^2(0, T; \mathscr{H})$$

into $\mathscr{W}$.

Proof. We follow the proof of Theorem 1.2 with some modifications. Let $F = L^2(0, T; D(\mathscr{A}(t)))$ with the norm

$$\| \mathscr{u} \|_F^2 = \int_0^T (| A_1(t)\, u(t)|^2 + | A_2(t)\, v(t)|^2)\, dt, \qquad \mathscr{u}(t) = [u(t), v(t)],$$

which by Lemma 3.1 is equivalent to the norm $\int_0^T \| \mathscr{A}(t)\, \mathscr{u}(t)\|^2\, dt$. We let

$$\Phi = \{\varphi \in F : \frac{d}{dt}(\mathscr{A}(t)\, \varphi(t)) \in L^2(0, T; \mathscr{H}),$$
$$\text{with } \varphi(0) \in \mathscr{H}_+(0) \text{ and } \varphi(T) \in \mathscr{H}_-(T)\}$$

with the norm

$$\| \varphi \|_\Phi^2 = \| \varphi \|_F^2 + ((\mathscr{A}(0)\, \varphi(0), \varphi(0))) - ((\mathscr{A}(T)\, \varphi(T), \varphi(T))).$$

Define $E(\mathscr{u}, \varphi) = \int_0^T ((\mathscr{A}\mathscr{u}, \mathscr{A}\varphi)) - ((\mathscr{u}, d/dt(\mathscr{A}\varphi)))\, dt$ on $F \times \Phi$. Then we have

$$-2 \operatorname{Re} \int_0^T ((\varphi, \frac{d}{dt}(\mathscr{A}\varphi)))\, dt = ((\mathscr{A}(0)\, \varphi(0), \varphi(0))) - ((\mathscr{A}(T)\, \varphi(T), \varphi(T)))$$
$$+ \int_0^T (\dot{A}_1^{-1} A_1 \varphi_1, A_1 \varphi_1) - (\dot{A}_2^{-1} A_2 \varphi_2, A_2 \varphi_2) dt,$$
$$\varphi(t) = [\varphi_1(t), \varphi_2(t)].$$

Hence

$$2 \operatorname{Re} E(\varphi, \varphi) \geqslant 2(1 - \alpha)(1 - \gamma) \int_0^T (| A_1 \varphi_1 |^2 + | A_2 \varphi_2 |^2)\, dt$$
$$+ ((\mathscr{A}(0)\, \varphi(0), \varphi(0))) - ((\mathscr{A}(T)\, \varphi(T), \varphi(T))).$$

The coercivity condition is satisfied and we may apply Theorem 1.1. Taking as a continuous linear form on Φ

$$L(\varphi) = \int_0^T ((\mathscr{f}, \mathscr{A}\varphi))\, dt + ((\mathscr{u}_0, \mathscr{A}(0)\, \varphi(0))) - ((\mathscr{u}_T, \mathscr{A}(T)\, \varphi(T))),$$

we know that there exists $\mathscr{u} \in F$ such that $E(\mathscr{u}, \varphi) = L(\varphi)$ for all $\varphi \in \Phi$. As in the proof of Theorem 1.2, it follows that

$$\frac{d\mathscr{u}}{dt} + \mathscr{A}\mathscr{u} = \mathscr{f} \qquad \text{in} \quad \mathscr{D}'(0, T; \mathscr{H}),$$

i.e., $\mathscr{u}(t) = [u(t), v(t)]$ satisfies the system (3.4). Thus $\mathscr{u}(t) \in \mathscr{W}$. Integrating by parts and substituting as in Theorem 1.2, one obtains

$$((\mathscr{u}(0) - \mathscr{u}_0\,, \mathscr{A}(0)\,\varphi(0))) = 0 \qquad \text{and} \qquad ((\mathscr{u}(T) - \mathscr{u}_T\,, \mathscr{A}(T)\,\varphi(T))) = 0$$

for all $\varphi \in \Phi$. It follows that $\mathscr{P}_+(0)\,\mathscr{u}(0) = \mathscr{u}_0$ and $\mathscr{P}_-(T)\,\mathscr{u}(T) = \mathscr{u}_T$.

For uniqueness of the solution, we apply the trace Lemma 2.3 to W_1 and W_2, whence $\mathscr{u} \to \mathscr{u}(0)$ is continuous from $\mathscr{W}$ into $D(A_1^{1/2}(0)) \oplus D(A_2^{1/2}(0))$, with the appropriate norms. $Q(t, \mathscr{u}) = ((\mathscr{A}(t)\mathscr{u}, \mathscr{u}))$, defined for $\mathscr{u} \in D(\mathscr{A}(t))$, extends continuously as

$$Q(t, \mathscr{u}) = \| \mathscr{A}_+^{1/2}(t)\,\mathscr{P}_+(t)\,\mathscr{u} \|^2 - \| \mathscr{A}_-^{1/2}(t)\,\mathscr{P}_-(t)\,\mathscr{u} \|^2$$

for $\mathscr{u} = [u, v] \in D(A_1^{1/2}(t) \oplus D(A_2^{1/2}(t))$. Now for $\mathscr{u}(t) = [u(t), v(t)] \in \mathscr{W}$, we have

$$\int_0^T \left(\left(\frac{d\mathscr{u}}{dt}, \mathscr{A}\mathscr{u}\right)\right) dt$$
$$= \int_0^T \left\{\left(\frac{du}{dt}, A_1 u\right) + \left(\frac{du}{dt}, v\right) + \left(\frac{dv}{dt}, u\right) - \left(\frac{dv}{dt}, A_2 v\right)\right\} dt.$$

Whence by the corollary to Lemma 2.3, we have

$$2\,\mathrm{Re} \int_0^T \left(\left(\frac{d\mathscr{u}}{dt}, \mathscr{A}\mathscr{u}\right)\right) dt = Q(T, \mathscr{u}(T)) - Q(0, \mathscr{u}(0))$$
$$- \int_0^T (\dot{A}_1^{-1} A_1 u, A_1 u)\, dt + \int_0^T (\dot{A}_2^{-1} A_2 v, A_2 v)\, dt.$$

Thus, if $d\mathscr{u}/dt + \mathscr{A}\mathscr{u} = 0$, we have

$$0 = 2 \int_0^T \| \mathscr{A}\mathscr{u} \|^2\, dt + 2\,\mathrm{Re} \int_0^T \left(\left(\frac{d\mathscr{u}}{dt}, \mathscr{A}\mathscr{u}\right)\right) dt$$
$$\geqslant 2(1 - \alpha)(1 - \gamma) \int_0^T (| A_1 u |^2 + | A_2 v |^2)\, dt + Q(T, \mathscr{u}(T)) - Q(0, \mathscr{u}(0)).$$

Hence if $\mathscr{P}_+(0)\,\mathscr{u}(0) = \mathscr{P}_-(T)\,\mathscr{u}(T) = 0$, we must have $\mathscr{u} = 0$. This completes the proof of Theorem 3.1.

While the subspaces $\mathscr{H}_+(0)$ and $\mathscr{H}_-(T)$ are perhaps the most "natural" for the operator $\mathscr{A}(t)$, we should like to solve the system (3.4) for initial and terminal conditions of the form $u(0) = u_0$ and $v(T) = v_T$, where u_0 and v_T are given in appropriate subspaces of H. We have not, as yet, explored this problem using only (3.1), (3.2), (3.3), and (3.6), but in the following special case, we have this result.

Suppose $t \to A(t)$ is a family of positive self-adjoint operators in H, $t \in [0, T]$. Consider the system

$$(3.7) \qquad \begin{aligned} &\frac{du}{dt}(t) + A(t)\,u(t) + v(t) = f(t), \\ &\frac{dv}{dt}(t) - A^{\rho}(t)\,v(t) + u(t) = g(t), \end{aligned}$$

where $A^{\rho}(t)$ is the fractional power of $A(t)$, $0 \leqslant \rho \leqslant 1$, defined via the operator calculus.

We shall suppose that

(3.8) $(A(t)u, u) \geqslant c \mid u \mid^2$, for all $u \in D(A(t))$, where $c > 1$ is a constant independent of t.

Then with $A_1(t) = A(t)$ and $A_2(t) = A^{\rho}(t)$, it follows that (3.1) and (3.2) are satisfied. Hence by Lemma 3.1 the matrix

$$\mathscr{A}(t) = \begin{bmatrix} A(t) & I \\ I & -A^{\rho}(t) \end{bmatrix}$$

is self-adjoint and invertible in $\mathscr{H} = H \oplus H$. Moreover, one can write explicitly

$$\mathscr{A}^{-1}(t) = \begin{bmatrix} A^{\rho}(t)(A^{\rho+1}(t) + I)^{-1} & (A^{\rho+1}(t) + I)^{-1} \\ (A^{\rho+1}(t) + I)^{-1} & -A(t)(A^{\rho+1}(t) + I)^{-1} \end{bmatrix}.$$

THEOREM 3.2. *Suppose that $t \to A(t)$ is a family of self-adjoint operators in H for $t \in [0, T]$ satisfying* (3.8). *Assume that $A_1(t) = A(t)$ and $A_2(t) = A^{\rho}(t)$ satisfy* (3.3) *and* (3.6) *with $\gamma = \frac{1}{2}(c^{-1} + c^{-\rho})$. Then for $u_0 \in D(A(0)^{(\rho+1)/2})$, $v_T \in D(A^{\rho}(T))$, and $\not{f} = [f, g] \in L^2(0, T; \mathscr{H})$, there is a unique solution $\not{u}(t) = [u(t), v(t)] \in \mathscr{W}$ satisfying the system* (3.7) *with $u(0) = u_0$ and $v(T) = v_T$.*

Proof. Let $K_+ = \{[u, 0] : u \in H\}$ and $K_- = \{[0, v] : v \in H\}$. Further, let $G_+ = \mathscr{A}^{-1}(0)K_+$ and $G_- = \mathscr{A}^{-1}(T)K$. Then for $\not{u} = \mathscr{A}^{-1}(0)\not{v}$, where $\not{v} = [\psi, 0] \in K_+$,

$$(3.9) \qquad \begin{aligned} ((\mathscr{A}(0)\not{u}, \not{u})) &= ((\not{v}, \mathscr{A}^{-1}(0)\not{v})) \\ &= (\psi, A^{\rho}(0)(A^{\rho+1}(0) + I)^{-1}\psi) \\ &\geqslant c^{-\rho/2} \mid (A^{\rho+1}(0) + I)^{-1/2}\psi \mid^2. \end{aligned}$$

Similarly, if $\not{u} = \mathscr{A}^{-1}(T)\not{v}$, where $\not{v} = [0, \xi] \in K_-$,

$$(3.10) \qquad -((\mathscr{A}(T)\not{u}, \not{u})) \geqslant c^{-\rho} \mid A^{(1-\rho)/2}(T)(A^{\rho+1}(T) + I)^{-1/2}\xi \mid^2.$$

With $F = L^2(0, T; D(\mathscr{A}(t)))$ as before in the proof of Theorem 3.1, we now take Φ as the space of functions $\varphi \in F$ such that

$$\frac{d}{dt}(\mathscr{A}(t)\,\varphi(t)) \in L^2(0, T; \mathscr{H}), \varphi(0) \in G_+ \qquad \text{and} \qquad \varphi(T) \in G_- .$$

By virtue of (3.9) and (3.10) we may define on Φ the norm

$$\| \varphi \|_\Phi^2 = \| \varphi \|_F^2 + ((\mathscr{A}(0)\,\varphi(0), \varphi(0))) - ((\mathscr{A}(T)\,\varphi(T), \varphi(T))).$$

Then as in Theorem 3.1, we set

$$E(u, \varphi) = \int_0^T ((\mathscr{A}u, \mathscr{A}\varphi)) - \left(\left(u, \frac{d}{dt}(\mathscr{A}\varphi)\right)\right) dt$$

and as before,

$$2 \operatorname{Re} E(u, \varphi) \geqslant 2(1-\alpha)(1-\gamma) \int_0^T (| Au |^2 + | A^\rho v |^2)\, dt$$
$$+ ((\mathscr{A}(0)\,\varphi(0), \varphi(0))) - ((\mathscr{A}(T)\,\varphi(T), \varphi(T))),$$

which satisfies the coercivity conditions. Now we set

$$L(\varphi) = \int_0^T ((f, \mathscr{A}\varphi))\, dt + (u_0, \psi) - (v_T, \xi),$$

where

$$\mathscr{A}(0)\,\varphi(0) = [\psi, 0] \in K_+$$

and

$$\mathscr{A}(T)\,\varphi(T) = [0, \xi] \in K_- .$$

We are assuming $u_0 \in D(A^{(\rho+1)/2}(0))$, so that

$$(u_0, \psi) = (A^{(\rho+1)/2}(0)u_0, A^{-(\rho+1)/2}(0)\psi)$$
$$\leqslant | A^{(\rho+1)/2}(0)u_0 | \, | A^{-(\rho+1)/2}(0)\psi |.$$

But

$$| A^{-(\rho+1)/2}(0)\psi |^2 \leqslant c_1 | (A^{\rho+1}(0) + I)^{-1/2}\psi |^2$$
$$\leqslant c_1 c^{\rho/2}((\mathscr{A}(0)\,\varphi(0), \varphi(0))), \qquad c_1 > 0 \text{ a constant},$$

by (3.9). Similarly, because $v_T \in D(A^\rho(T))$,

$$(v_T, \xi) \leqslant | A^\rho(T)v_T | \, | A^{-\rho}(T)\xi |$$

and

$$| A^{-\rho}(T)\xi |^2 \leqslant -c_2 c^\rho((\mathscr{A}(T)\,\varphi(T), \varphi(T))), \qquad c_2 > 0 \text{ a constant}.$$

It follows that $L(\varphi)$ is continuous on Φ. Thus, by Theorem 1.1, there is a function $\mathscr{u} \in F$ such that $E(\mathscr{u}, \varphi) = L(\varphi)$ for all $\varphi \in \Phi$. This implies that $\mathscr{u} \in \mathscr{W}$, and integrating by parts as in Theorem 1.2, one finds

$$((\mathscr{u}(0), \mathscr{A}(0)\,\varphi(0))) = (([u_0, 0], \mathscr{A}(0)\,\varphi(0)))$$

and

$$((\mathscr{u}(T), \mathscr{A}(T)\,\varphi(T))) = (([0, v_T], \mathscr{A}(T)\,\varphi(T)))$$

for all $\varphi \in \Phi$. This implies that $(u(0), \psi) = (u_0, \psi)$ for all $\psi \in H$ and $(v(T), \xi) = (v_T, \xi)$ for all $\xi \in H$. Hence $u(0) = u_0$ and $v(T) = v_T$.

It remains to show that this solution is unique. As in Theorem 3.1,

$$Q(t, \mathscr{v}) = ((\mathscr{A}(t)\mathscr{v}, \mathscr{v})) \qquad \text{defined for} \quad \mathscr{v} \in D(\mathscr{A}(t))$$

extends continuously to $D(A^{1/2}(t)) \times D(A^{\rho/2}(t))$ as

$$Q(t, \mathscr{v}) = \| \mathscr{A}_+^{1/2}(t)\, \mathscr{P}_+(t)\mathscr{v} \|^2 - \| \mathscr{A}_-(t)\, \mathscr{P}_-(t)\mathscr{v} \|^2.$$

Now for $\mathscr{u} = [0, \xi] \in D(\mathscr{A}(0)) \cap K_-$ and $\mathscr{v} = [\psi, 0] \in D(\mathscr{A}(T)) \cap K_+$ we have $((\mathscr{A}(0)\mathscr{u}, \mathscr{u})) = -(A^\rho(0)\xi, \xi) \leqslant 0$ and $((\mathscr{A}(T)\mathscr{v}, \mathscr{v})) = (A(T)\psi, \psi) \geqslant 0$. Thus

$$\begin{aligned} &Q(0, \mathscr{u}) \leqslant 0 \qquad \text{for} \quad \mathscr{u} = [0, \xi], \quad \xi \in D(A^{\rho/2}(0)), \\ &Q(T, \mathscr{v}) \geqslant 0 \qquad \text{for} \quad \mathscr{v} = [\psi, 0], \quad \psi \in D(A^{1/2}(0)). \end{aligned} \tag{3.11}$$

Now if $\mathscr{u}(t) = [u(t), v(t)]$ is a solution of the system (3.8) with $\mathscr{f}(t) = [f(t), g(t)] = 0$ and $u(0) = v(T) = 0$, then as in Theorem 3.1, multiplying the equation $d\mathscr{u}/dt + \mathscr{A}\mathscr{u} = 0$ by $\mathscr{A}\mathscr{u}$ and integrating, we obtain

$$\begin{aligned} 0 \geqslant 2(1 - \alpha)(1 - \gamma) \int_0^T (| Au |^2 + | A^\rho v |^2)\, dt \\ + Q(T, \mathscr{u}(T)) - Q(0, \mathscr{u}(0)). \end{aligned}$$

But (3.11) implies that when $u(0) = v(T) = 0$, we have

$$Q(T, \mathscr{u}(T)) - Q(0, \mathscr{u}(0)) \geqslant 0,$$

whence $\mathscr{u} = 0$. This completes the proof of Theorem 3.2.

We remark that if one does not require the solutions to (3.4) to be in $W_1 \oplus W_2$, then one may weaken the hypotheses considerably. In particular, one need no longer require the operators $A_1(t)$ and $A_2(t)$ to be self-adjoint, nor need they satisfy the conditions (3.2), (3.3), and (3.6).

For instance, suppose that V_1 and V_2 are two dense subspaces of H, with Hilbert norms $\|\cdot\|_1$ and $\|\cdot\|_2$ finer than $|\cdot|_H$. Let $a_1(t, u, v)$ and $a_2(t, u, v)$ be two families of continuous sesquilinear forms on V_1 and V_2, $0 \leqslant t \leqslant T$. Let $A_1(t)$ and $A_2(t)$ be the unbounded operators in H such that $a_i(t, u, v) = (A_i(t)u, v)$, $u \in D(A_i(t))$, $v \in V_i$ $(i = 1, 2)$. Suppose that for $u, v \in V_i$, $t \to a_i(t, u, v)$ $(i = 1, 2)$ is measurable; $\operatorname{Re} a_i(t, u, u) \geqslant c_i \| u \|_i^2$, $u \in V_i$, $0 \leqslant t \leqslant T$; and $| a_i(t, u, v)| \leqslant \gamma_i \| u \|_i \| v \|_i$, $u, v \in V_i$, $0 \leqslant t \leqslant T$. Then for $u_0 \in H$ and $v_T \in H$, f and $g \in L^2(0, T; H)$, there are "weak solutions" to Eq. (3.4) such that

$$u \in L^2(0, T; V_1) \quad \text{and} \quad u' \in L^2(0, T; V_1'),$$

$$v \in L^2(0, T; V_2) \quad \text{and} \quad v' \in L^2(0, T; V_2'),$$

and

$$u(0) = u_0,$$

$$v(T) = v_T.$$

Results of this type are discussed in Refs. [14, 15].

Before leaving these systems of evolution equations we wish to point out their connection with control theory. Systems of the type (3.7) with $\rho = 1$ arise in the following situation. Let $A(t)$ be a family of positive self-adjoint operators in a separable Hilbert space H, with domain $D(A(t))$. Suppose that $A(t)$ satisfies (1.4), (1.5) and

$$(3.12) \quad \left| \frac{d}{dt} (A^{-1}(t)f, f) \right| \leqslant 2\alpha\, | f |^2 + c\, | A^{-1/2}(t) f |_H^2, \qquad 0 \leqslant \alpha < 1,$$

and $c > 0$ constants.

Let $y_0 \in D(A^{1/2}(0))$ and $f(t) \in L^2(0, T; H)$ be given functions. Then for each control function $u \in L^2(0, T; H)$ there is a unique solution $y \in W = L^2(0, T; D(A(t))) \cap H^1(0, T; H)$ to the Cauchy problem (see Lions, Ref. [14])

$$y' + A(t)y = f + u,$$

$$y(0) = y_0.$$

Let the "cost function" for this system, whose state is y, be given by

$$J(u) = \int_0^T | y(t, u) - z(t)|^2 \, dt + \int_0^T | u(t)|^2 \, dt,$$

where $z(t) \in L^2(0, T; H)$ is the desired state. It is known that in these circumstances, the problem of finding an optimal control $u_0 \in L^2(0, T; H)$

which minimizes $J(u)$ has a unique solution (see Lions and Magenes, Ref. [15, Vol. II, Chap. 8]). In characterizing the optimal control, one shows that the system

$$\begin{cases} y' + A(t)y + p = f, \\ p' - A(t)p + y = z, \end{cases}$$

$$\begin{cases} y(0) = y_0 \in D(A^{1/2}(0)), \\ p(T) = 0 \end{cases}$$

has a unique solution pair $[y, p] \in W \oplus W$, and that the optimal control is just $u_0 = -p$. Thus, in this case ($\rho = 1$), with the condition $p(T) = 0$, one is able to solve the system (3.7) for $y_0 \in D(A^{1/2}(0))$, under the hypothesis (3.12), which is less restrictive than (3.3) and (3.6).

4. We now turn to the problem of applying the Lions–Malgrange backward uniqueness theorem to the solutions of problem $(*)$. For completeness, we give the proof, following Ref. [16].

A different hypothesis must be made on the weak derivative of $A^{-1}(t)$. Assume that $A(t)$ satisfies (1.4) and (1.5) and that for all $f \in H$ and all ϵ, $0 < \epsilon \leqslant 1$,

$$\left| \frac{d}{dt} (A^{-1}(t) f, f) \right| \leqslant \epsilon \, | f |^2 + c_1 \epsilon^{-1} \, | A^{-1}(t) f |^2, \tag{4.1}$$

where $c_1 > 0$ is a constant independent of $t \in [0, T]$.

Lemma 4.1. *Assume $A(t)$ satisfies* (1.4), (1.5), *and* (4.1). *Suppose*

$$s \leqslant \min\left(T, \frac{1}{4\sqrt{c_1}}\right) \quad \text{and} \quad s \leqslant r \leqslant r + s \leqslant T.$$

Let $u \in W(r, r+s)$ such that $u(r) = u(r+s) = 0$. Then for $k \geqslant 2c_1$,

$$\int_r^{r+s} e^{k(t-r)^2} \, | Au + u' |^2 \, dt \geqslant \frac{k}{4} \int_r^{r+s} e^{k(t-r)^2} \, | u |^2 \, dt. \tag{4.2}$$

Proof. For simplicity we set $r = 0$. This will not affect the generality. We set $w = e^{+kt^2/2} u$. Then to prove (4.2) it suffices to show

$$\int_0^s | Aw - ktw + w' |^2 \, dt \geqslant \frac{k}{4} \int_0^s | w |^2 \, dt.$$

Putting

$$X_k = \int_0^s (| Aw - ktw |^2 + | w' |^2) \, dt$$

and

$$Y_k = 2 \operatorname{Re} \int_0^s (Aw - ktw, w')\, dt,$$

we must show

$$X_k + Y_k \geqslant \frac{k}{4} \int_0^s |w|^2\, dt.$$

To evaluate Y_k, note first that if $w \in W_0(0, s)$ and $w(0) = w(s) = 0$, then integration by parts yields

$$2 \operatorname{Re} \int_0^s (Aw, w')\, dt = \int_0^s (\dot{A}^{-1}v, v)\, dt, \qquad v = Aw.$$

By a procedure similar to that used in Lemma 2.2, it can be shown that these functions are dense in the space of $w \in W(0, s)$ such that $w(0) = w(s) = 0$, and the equality extends to this latter class. Then using (4.1) we have

$$Y_k \geqslant (k - c_1\epsilon^{-1}) \int_0^s |w|^2\, dt - \epsilon \int_0^s |Aw|^2\, dt. \tag{4.3}$$

Hence

$$X_k + Y_k \geqslant \int_0^s (k - c_1\epsilon^{-1} - 2\epsilon k^2t^2)\, |w|^2\, dt + (1 - 2\epsilon) \int_0^s |Aw - ktw|^2\, dt.$$

Now ϵ may be chosen, $0 < \epsilon \leqslant 1$, so we take $\epsilon = 2c_1/k$. Then

$$k - c_1\epsilon^{-1} - 2\epsilon k^2 t^2 = k(\tfrac{1}{2} - 4c_1t^2),$$

which implies

$$X_k + Y_k \geqslant \left(1 - \frac{2c_1}{k}\right) \int_0^s |Aw - kt|^2\, dt + k \int_0^s \left(\frac{1}{2} - 4c_1t^2\right) |w|^2\, dt.$$

For $0 \leqslant t \leqslant s \leqslant 1/(4\sqrt{c_1})$, we have $\frac{1}{2} - 4c_1t^2 \geqslant \frac{1}{4}$, so for $k \geqslant 2c_1$ it follows that

$$X_k + Y_k \geqslant \frac{k}{4} \int_0^s |w|^2\, dt.$$

This completes the proof of the Lemma.

THEOREM 4.1 (Lions–Malgrange). *Assume* $A(t)$ *satisfies* (1.4), (1.5), *and* (4.1). *Let* $u \in W$ *such that*

(i) $Au + u' = 0$;

(ii) $u(T) = 0$.

Then $u = 0$.

Proof. Let s be chosen as in Lemma 3.1. Let $q(t)$ be a real-valued $\mathscr{C}^\infty$ function such that

$$q(t) = \begin{cases} 1 & \text{for } t \in [T - s/2, T], \\ 0 & \text{for } t \in [0, T - s]. \end{cases}$$

We set $\varphi = qu$. Now $\varphi \in W$ and $\varphi(T - s) = \varphi(T) = 0$, so that we may apply Lemma 4.1:

$$\int_{T-s}^{T} e^{k(t-(T-s))^2} \mid A\varphi + \varphi' \mid^2 dt \geqslant \frac{k}{4} \int_{T-s}^{T} e^{k(t-(T-s))^2} \mid \varphi \mid^2 dt \tag{4.4}$$

for $k \geqslant 2c_1$. Since $A\varphi + \varphi' = qu$ and $q' = 0$ in $[T - s/2, T]$ we have

$$\begin{aligned}\int_{T-s}^{T} e^{k(t-(T-s))^2} \mid A\varphi + \varphi' \mid^2 dt &\leqslant e^{ks^2/4} \int_{T-s}^{T-s/2} \mid q' \mid^2 \mid u \mid^2 dt \\ &\leqslant c_2 e^{ks^2/4},\end{aligned}$$

where $c_2 > 0$ is constant. Considering the right side of (4.4) we have

$$\frac{k}{4} \int_{T-s}^{T} e^{k(t-(T-s))^2} \mid \varphi \mid^2 dt \geqslant \frac{k}{4} e^{ks^2/4} \int_{T-s/2}^{T} \mid u \mid^2 dt.$$

Thus (4.4) becomes

$$\frac{k}{4} \int_{T-s/2}^{T} \mid u \mid^2 dt \leqslant c_2 \qquad \text{for all} \quad k \geqslant 2c_1,$$

and this implies $u = 0$ on $[T - s/2, T]$. By repeating this argument a finite number of times, one can show $u = 0$ on $[0, T]$, thus proving the theorem.

We note that it was not necessary to assume that $A(t) \geqslant 0$, nor was it necessary to introduce the hypothesis (2.1) on $B(t)$. Furthermore, it is evident that the following slightly more general hypothesis may be used instead of (4.1). Assume that $A(t)$ satisfies (1.4) and (1.5), and that there is a real number k such that

(4.1)′ $A(t) + kI$ satisfies (1.4), (1.5), (4.1).

COROLLARY. *Suppose that* $A(t)$ *satisfies* (1.4), (1.5), *and* (4.1)$'$ *on* $[0, T]$. *Let* $u(t) \in W$ *be a solution of* $Au + u' = 0$ *and suppose that for some* $\tau \in [0, T]$, $u(\tau) = 0$. *Then* $u(t) = 0$ *on all of* $[0, T]$.

Proof. Applying Theorem 4.1 directly, we immediately obtain $u = 0$ on $[0, \tau]$. Now set $v(s) = u(T - s)$. Then $v'(s) - A(T - s)\, v(s) = 0$ and $v(T - \tau) = u(\tau) = 0$. However, $-A(T - s)$ still satisfies (1.4), (1.5), and (4.1)$'$, and we deduce that $v = 0$ on $[0, T - \tau]$, whence $u = 0$ on $[\tau, T]$ also.

Now we apply this result to the solutions of the two-point problem obtained in Theorem 1.2.

THEOREM 4.2. *Assume that* $A(t)$ *satisfies* (1.4), (1.5), (1.6), (2.1), *and* (4.1)$'$. *Let* $u \in W$ *be a nontrivial solution of* $Au + u' = 0$. *Then* $Q(t, u(t))$ *is monotone decreasing on* $[0, T]$. *In fact, if* $s, t \in [0, T]$ *with* $s < t$, *we have*

$$Q(t, u(t)) < Q(s, u(s)). \tag{4.5}$$

Proof. Suppose for some t, s with $0 \leqslant s < t \leqslant T$, that (4.5) does not hold. Then multiplying and integrating on $[s, t]$ we find

$$\begin{aligned} 0 &= 2 \int_s^t |\, Au\,|^2\, d\sigma - 2 \operatorname{Re} \int_s^t (u', Au)\, d\sigma \\ &= 2 \int_s^t |\, Au\,|^2\, d\sigma + \int_s^t (\dot{A}^{-1} Au, Au)\, d\sigma + Q(t, u(t)) - Q(s, u(s)) \\ &\geqslant 2(1 - \alpha) \int_s^t |\, Au\,|^2\, d\sigma, \end{aligned}$$

using (1.6) and (2.2). Thus $u = 0$ on $[s, t]$, whence $u = 0$ on $[0, T]$ by the corollary to Theorem 4.1.

COROLLARY. *If* $u \in W$ *is a solution of* $Au + u' = 0$, *and* $u_T = 0$, *then*

$$\max_{0 \leqslant t \leqslant T} |\, u(t)| = |\, u(0)| \qquad \text{and} \qquad \min_{0 \leqslant t \leqslant T} |\, u(t)| = |\, u(T)|.$$

Proof. Since $P(T)\, u(T) = 0$, we have, by Theorem 4.2, for $0 \leqslant t \leqslant T$,

$$Q(t, u(t)) \geqslant Q(T, u(T)) = |\, A_+^{1/2}(T)\, u(T)|^2 > 0.$$

Multiplying $Au + u' = 0$ by u and integrating on $[s, t]$, $s < t$,

$$-\int_s^t (u', u)\, d\sigma = \int_s^t (Au, u)\, d\sigma = \int_s^t Q(s, u(s))\, d\sigma.$$

Thus,

$$| u(s)|^2 - | u(t)|^2 = -2 \operatorname{Re} \int_s^t (u', u) \, d\sigma \geqslant 0$$

for each $s, t \in [0, T]$, with $s < t$. Since $t \to u(t)$ is continuous with values in H, the max and min are attained and these clearly must be $| u(0)|$ and $| u(T)|$.

Finally we note that Agmon and Nirenberg have also proved a backward uniqueness theorem for a family of operators in which $A(t)$ is not assumed positive. Their article (see Ref. [1]) however, considers solutions u which are continuously differentiable in H strongly, and for which $A(t)u$ is strongly continuous in H.

A slight modification of the proof of Lemma 4.1 will allow Theorem 4.1 to be applied to the systems considered in Section 3. In fact, suppose that $A_1(t)$ and $A_2(t)$ are two families of positive self-adjoint operators which satisfy (3.1), (3.2), and (3.3). Suppose in addition that both $A_1(t)$ and $A_2(t)$ satisfy (4.1) with the same constant $c_1 > 0$. Let $\mathscr{A}(t)$ be the matrix operator

$$\mathscr{A}(t) = \begin{bmatrix} A_1(t) & I \\ I & -A_2(t) \end{bmatrix}.$$

Now using the notation of Section 3, let $\omega \in \mathscr{W}$ be a solution of $\mathscr{A}\omega + \omega' = 0$ on $[0, s]$ with $\omega(0) = \omega(s) = 0$. The crucial inequality in Lemma 4.1 is (4.3), and this may be obtained by observing that

$$\begin{aligned} &2 \operatorname{Re} \int_0^s ((\mathscr{A}\omega, \omega')) \, dt \\ &\quad = \int_0^s (v_1, \dot{A}_1^{-1} v_1) \, dt - \int_0^s (v_2, \dot{A}_2^{-1} v_2) \, dt, \qquad \omega(t) = [w_1(t), w_2(t)], \end{aligned}$$

where $v_1(t) = A_1(t) \, w_1(t)$ and $v_2(t) = A_2(t) \, w_2(t)$. But then by Lemma 3.1 and the assumption that $A_1(t)$ and $A_2(t)$ satisfy (4.1),

$$\begin{aligned} &2 \operatorname{Re} \int_0^s ((\mathscr{A}\omega, \omega')) \, dt \\ &\geqslant -\epsilon \int_0^s (| v_1 |^2 + | v_2 |^2) \, dt - c_1 \epsilon^{-1} \int_0^s (| w_1 |^2 + | w_2 |^2) \, dt \\ &\geqslant -c\epsilon \int_0^s \| \mathscr{A}\omega \|^2 \, dt - c_1 \epsilon^{-1} \int^s \| \omega \|^2 \, dt \end{aligned}$$

and (4.3) follows immediately.

A similar modification of the proof of Theorem 4.2 will show that for $0 \leqslant s < t \leqslant T$,

$$Q(t, u(t)) < Q(s, u(s)),$$

where $u(t) \in \mathscr{W}$ is a solution of $\mathscr{A}u + u' = 0$ on $[0, T]$.

5. In this last section we shall present examples of self-adjoint operators $A(t)$ which satisfy (1.4), (1.5), (1.6), (2.1), and (4.1)′ needed for the theorems of earlier sections. For convenience we recall those hypotheses here.

(1.4) For each $t \in [0, T]$, $A^{-1}(t)$ exists and is a continuous operator on H.

(1.5) For $f, g \in H$, the function $t \to (A^{-1}(t)f, g)$ is continuously differentiable on $[0, T]$.

(1.6) $| d/dt(A^{-1}(t)f, g)| \leqslant 2\alpha \,| f |_H^2$, $f \in H$, where $0 \leqslant \alpha < 1$ is a constant.

(2.1) $B(t)$, the "absolute value" of $A(t)$, also satisfies (1.5).

(4.1)′ For some $k \in \mathbf{R}$, $A(t) + kI$ satisfies (1.4), (1.5) and there is a constant $c > 0$ such that

$$\left| \frac{d}{dt}((A(t) + kI)^{-1}f, f) \right| \leqslant \epsilon\, | f |^2 + c\epsilon^{-1}\, |(A(t) + kI)^{-1}f\,|^2$$

for all ϵ, $0 < \epsilon \leqslant 1$.

We first prove a lemma which will be used several times in this section.

LEMMA 5.1. *For $t \in [0, T]$, let $M(t)$ be a family of self-adjoint operators in H satisfying* (1.4) *and* (1.5). *Let $\sigma(M(t))$ and $\sigma(M^{-1}(t))$ denote the spectrum of $M(t)$ and $M^{-1}(t)$, respectively. Suppose that $\lambda \in \mathbf{C}$ such that $\lambda \notin \sigma(M(t))$ for all $t \in [0, T]$. Then $R_\lambda(t) = (\lambda I - M(t))^{-1}$ satisfies* (1.5) *with*

$$\frac{d}{dt}(R_\lambda(t)f, g) = (\dot{R}_\lambda(t)f, g), \qquad f, g \in H,$$

where $\dot{R}_\lambda(t) = -M(t)\, R_\lambda(t)\, \dot{M}^{-1}(t)\, M(t)\, R_\lambda(t)$.

Proof. We may assume $\lambda \neq 0$, and it follows, by a version of the spectral mapping theorem (see Dunford and Schwartz, Ref. [10]), that $1/\lambda \notin \sigma(M(t)^{-1})$. By Lemma 1.1, $t \to M^{-1}(t)$ is continuous in the uniform operator norm in $\mathscr{L}(H, H)$, as is $t \to M^{-1}(t) - (1/\lambda)I$. Because the operation

$A \to A^{-1}$ is continuous in $\mathscr{L}(H, H)$ for invertible elements, it follows that $t \to (M^{-1}(t) - (1/\lambda)I)^{-1}$ is also continuous in the uniform norm. Thus there exists $\rho > 0$ such that $d(1/\lambda, \sigma(M^{-1}(t))) \geqslant \rho > 0$ for all $t \in [0, T]$.

Now let $h(x) = (x\lambda - 1)^{-1}$ for $x \in \mathbf{R}$. Then $h(x)$ is continuous on the bounded set

$$S = \bigcup_{0 \leqslant t \leqslant T} \sigma(M^{-1}(t)).$$

We shall now show that $M(t)\, R_\lambda(t) = h(M^{-1}(t))$ is also continuous in the uniform norm. In fact, let $p_n(x)$ be a sequence of polynomials which converge uniformly to h on S. Then for each n, $t \to p_n(M^{-1}(t))$ is continuous in the uniform norm and

$$\| p_n(M^{-1}(t)) - h(M^{-1}(t))\| \leqslant \sup_S | p_n(x) - h(x)|$$

by the Gelfand Theorem (see Dunford and Schwarz, Ref. [10]). Thus $h(M^{-1}(t))$, being the uniform limit of $p_n(M^{-1}(t))$, must be continuous. Now for $t, s \in [0, T]$,

$$\begin{aligned} R_\lambda(t) - R_\lambda(s) &= R_\lambda(s)(\lambda I - M(s))\, R_\lambda(t) - R_\lambda(s)(\lambda I - M(t))\, R_\lambda(t) \\ &= M(s)\, R_\lambda(s)(\lambda M^{-1}(s)\, M^{-1}(t) - M^{-1}(t))\, M(t)\, R_\lambda(t) \\ &\quad - M(s)\, R_\lambda(s)(\lambda M^{-1}(s)\, M^{-1}(t) - M^{-1}(s))\, M(t)\, R_\lambda(t) \\ &= M(s)\, R_\lambda(s)(M^{-1}(s) - M^{-1}(t))\, M(t)\, R_\lambda(t). \end{aligned}$$

Then for $f, g \in H$,

$$\begin{aligned} &\frac{1}{t-s}((R_\lambda(t) - R_\lambda(s))f, g) \\ &\qquad \frac{1}{t-s}(M(s)\, R_\lambda(s)(M^{-1}(s) - M^{-1}(t))\, M(t)\, R_\lambda(t)f, g) \\ &\qquad = \frac{1}{t-s}((M^{-1}(s) - M^{-1}(t))\, M(t)\, R_\lambda(t)f, M(s)\, R_{\bar{\lambda}}(s)\, g). \end{aligned}$$

Therefore

$$\frac{d}{dt}(R_\lambda(t)f, g) = \lim_{s \to t} \frac{1}{t-s}((R_\lambda(t) - R_\lambda(s))f, g)$$

exists and

$$t \to (\dot{R}_\lambda(t)f, g) = -(\dot{M}^{-1}(t)\, M(t)\, R_\lambda(t)f, M(t)\, R_{\bar{\lambda}}(t) g)$$

is continuous, thus proving the lemma.

We are now in a position to give a fairly large class of operators $A(t)$ which satisfy (1.4), (1.5), (1.6), and (4.1)'. With H as usual a separable Hilbert space with norm $|f|$, suppose that V is another separable Hilbert space with norm $\|v\|$, such that

(5.1) $V \subset H$ with $|v| \leqslant \|v\|$ for $v \in V$, and V dense in H.

For $t \in [0, T]$, let $a(t, u, v)$ be a family of continuous sesquilinear forms on V such that

(5.2) For each $u, v \in V$, $t \to a(t, u, v)$ is continuously differentiable on $[0, T]$.

(5.3) $a(t, u, v) = \overline{a(t, v, u)}$ for all $u, v \in V$, and there exists $\lambda_0 > 0$ such that for all $t \in [0, T]$ and $u \in V$,

$$a(t, u, u) + \lambda_0 |u|^2 \geqslant c \|u\|_V^2, \qquad c > 0 \text{ a constant.}$$

Now for each $t \in [0, T]$, the form $a(t, u, v)$ and the spaces V and H determine a self-adjoint operator $A(t)$ with $D(A(t)) \subset V$ and such that for $u \in D(A(t))$ and $v \in V$,

$$a(t, u, v) = (A(t)u, v).$$

THEOREM 5.1. *Let $A(t)$, $t \in [0, T]$, be the family of self-adjoint operators defined by the space V and the forms $a(t, u, v)$ satisfying* (5.1), (5.2), *and* (5.3). *Suppose that $A(t)$ satisfies* (1.4). *Then $A(t)$ satisfies* (1.5) *and* (4.1)'.

Proof. From (5.3) it is clear that $A(t) + \lambda_0 I > 0$, $0 \leqslant t \leqslant T$. Let $f \in H$, and set $u(t) = (A(t) + \lambda_0 I)^{-1} f \in V$. Then $((A(t) + \lambda_0 I)^{-1} f, f) = a(t, u(t), u(t)) + \lambda_0 |u(t)|^2$. It follows (see Lions, Ref. [14, Chap. VII]) that

$$\frac{d}{dt}((A(t) + \lambda_0 I)^{-1} f, f) = -a'(t, u(t), u(t)),$$

where $a'(t, u, v) = (d/dt)\, a(t, u, v)$ for $u, v \in V$. Thus $A(t) + \lambda_0 I$ satisfies (1.4) and (1.5). The Banach–Steinhaus Theorem and (5.3) imply that there is a constant $c_1 > 0$ such that

$$|a'(t, u, v)| \leqslant c_1 \|u\| \|v\| \qquad \text{for} \quad u, v \in V \text{ and } 0 \leqslant t \leqslant T.$$

Hence by (5.3),

$$\begin{aligned} |a'(t, u(t), u(t))| &\leqslant c_1 \|u(t)\|^2 \leqslant c_1 c^{-1}[a(t, u(t), u(t)) + \lambda_0 |u(t)|^2] \\ &\leqslant c_1 c^{-1}((A(t) + \lambda_0 I)^{-1} f, f) \\ &\leqslant \epsilon |f|^2 + \epsilon^{-1} \left(\frac{c_1}{2c}\right)^2 |(A(t) + \lambda_0 I)^{-1} f|^2 \end{aligned}$$

for all $\epsilon > 0$. Hence (4.1)' is satisfied.

Setting $M(t) = A(t) + \lambda_0 I$, and applying Lemma 5.1, we have that $A(t)$ also satisfies (1.5). This proves Theorem 5.1.

We shall see later in Theorem 5.3 that if $A(t)$ satisfies (1.4), (1.5), and an additional hypothesis, then $A(t)$ satisfies (2.1). However, in the case that $A(t)$ is semibounded from below uniformly on $[0, T]$, as it is in Theorem 5.1, this is not necessary. In fact, the trace Lemma 2.3 and the resulting uniqueness theorem only use the following more general hypothesis:

(2.1)′ There exists a positive self-adjoint operator $\Lambda(t)$ such that $D(\Lambda(t)) = D(A(t))$ for all $t \in [0, T]$, $c_1 \mid \Lambda(t)u \mid \leqslant \mid A(t)u \mid \leqslant c_2 \mid \Lambda(t)u \mid$ for $u \in D(A(t))$, and $t \to (\Lambda^{-1}(t)f, g) \in \mathscr{C}^1[0, T]$ for all $f, g \in H$.

In the situation of Theorem 5.1, we may take

$$\Lambda(t) = A(t) + \lambda_0 I.$$

Finally we note that if $A(t)$ satisfies (1.5), then for $r > 0$, a constant sufficiently large, $rA(t)$ will satisfy (1.6).

Thus Theorem 5.1 yields a large class of families of operators for which the hypotheses of Sections 1, 2, and 4 are satisfied.

Theorem 5.1 also provides us with operators which satisfy the hypotheses (3.1), (3.2), (3.3), and (3.6) of Section 3. Indeed, suppose that V_1 and V_2, are two dense subspaces of H satisfying (5.1) and that $a_1(t, u, v)$ and $a_2(t, u, v)$] $t \in [0, T]$, are two families of continuous sesquilinear forms on V_1 and V_2, respectively, which satisfy (5.2) and (5.3) with $\lambda_0 = 0$. Let $A_1(t)$ and $A_2(t)$ be the self-adjoint operators associated with $a_1(t, u, v)$ and V_1, and $a_2(t, u, v)$ and V_2. Then there are constants c_1 and c_2 such that

$$(A_1(t)u, u) \geqslant c_1 \mid u \mid^2, \qquad u \in D(A_1(t)),\ t \in [0, T],$$

$$(A_2(t)v, v) \geqslant c_2 \mid v \mid^2, \qquad v \in D(A_2(t)),\ t \in [0, T].$$

Let $r > 0$ be a constant. Then for r sufficiently large the operators $rA_1(t)$ and $rA_2(t)$ will satisfy (3.2) and (3.6).

We now wish to demonstrate the existence of self-adjoint operators $A(t)$ which satisfy the hypotheses (1.4), (1.5), (1.6), and (2.1), and which are not uniformly semibounded from below as in Theorem 5.1.

First we show how from two families of operators satisfying (1.4), (1.5), and (2.1), one may construct a third using tensor products. Let E and F be separable Hilbert spaces. On $E \otimes F$ we define the scalar product

$$(u, v)_{E \otimes F} = \left(\sum_{i=1}^{n} x_i \otimes y_i , \sum_{j=1}^{m} z_j \otimes w_j \right)$$

$$= \sum_{i,j} (x_i , z_j)_E \, (y_i , w_j)_F ,$$

where

$$u = \sum_{i=1}^{n} x_i \otimes y_i \quad \text{and} \quad v = \sum_{j=1}^{n} z_j \otimes w_j \in E \otimes F.$$

For $u \in E \otimes F$ we set $\| u \| = (u, u)_{E\otimes F}^{1/2}$. Since $\| x \otimes y \| = | x |_E | y |_F$, this defines a cross norm on $E \otimes F$ and we set $H = E \widehat{\otimes} F$, the completion of $E \otimes F$, in this norm. $E \widehat{\otimes} F$ is a Hilbert space (see Dixmier, Ref. [9]).

Now for each $t \in [0, T]$, let $A_1(t)$ and $A_2(t)$ be self-adjoint operators in E and F, respectively. Define the operator $A_1(t) \otimes A_2(t)$ with domain $D(A_1(t)) \otimes D(A_2(t))$ dense in H. In general $A_1(t) \otimes A_2(t)$ will not be closed and we denote its closure by $A(t) = A_1(t) \widehat{\otimes} A_2(t)$.

THEOREM 5.2. *Suppose that $A_1(t)$ and $A_2(t)$ are two families of self-adjoint operators in E and F, respectively, which satisfy* (1.4), (1.5), *and* (2.1). *Then $A(t) = A_1(t) \widehat{\otimes} A_2(t)$ is self-adjoint in H and satisfies* (1.4), (1.5), *and* (2.1).

Proof. For $u = \sum_{i=1}^{n} x_i \otimes y_i$ and $v = \sum_{j=1}^{m} z_j \otimes w_j$, both in $D(A_1(t)) \otimes D(A_2(t))$, we have

$$\begin{aligned}(A(t)u, v)_H &= \sum_{i,j} (A_1(t)\, x_i , z_j)_E\, (A_2(t)\, y_i , w_j)_F \\ &= \sum_{i,j} (x_i , A_1(t)\, z_j)_E\, (y_i , A_2(t)\, w_j)_F \\ &= (u, A(t)v)_H .\end{aligned}$$

Thus $A_1(t) \otimes A_2(t) \subset A(t)^*$, the Hilbert adjoint of $A(t)$, whence $A(t) = A_1(t) \widehat{\otimes} A_2(t)) \subset A(t)^*$. To show the reverse inclusion, let $v \in D(A(t)^*)$. Then there is a sequence $v_n \in D(A_1(t)) \otimes D(A_2(t))$ such that $v_n \to v$ in H. Hence for each $u \in D(A_1(t)) \otimes D(A_2(t))$,

$$(A(t)u, v) = \lim_{n\to\infty} (u, A(t)\, v_n).$$

But $v \in D(A(t)^*)$ implies that $|(A(t)u, v)| \leqslant M \mid u |_H$, where $M > 0$ is a constant, perhaps depending on t. Hence by the Banach–Steinhaus Theorem $A(t)v_n$ converges weakly to some $f \in H$. That is, $A(t)^*$ is the weak closure of the graph of $A_1(t) \otimes A_2(t)$ in $H \times H$, and this is equal to the strong closure of the graph of $A_1(t) \otimes A_2(t)$. But this is precisely the graph of $A(t)$, and thus $A(t)$ is self-adjoint.

Next we show $A(t)$ is invertible in H. First recall that by Lemma 1.1 there are constants c_1 and $c_2 > 0$ such that for all $t \in [0, T]$ and $x \in D(A_1(t))$, $| A_1(t)x | \geqslant c_1 | x |$, and similarly for $A_2(t)$. Thus if $u = \sum x_i \otimes y_i$, where

$x_i \in D(A_1(t))$, and $y_i \in F$ (the y_i may be assumed pairwise orthogonal), we have

$$\|(A_1(t) \otimes I)u\|_H^2 = \sum_{i=1}^{n} | A_1(t)\, x_i |^2 \, | y_i |^2 \geqslant c_1{}^2 \| u \|_H^2 .$$

Similarly, for $v \in E \otimes D(A_2(t))$ we have

$$\|(I \otimes A_2(t))v\|^2 \geqslant c_2{}^2 \| v \|_H^2 .$$

Then for $u \in D(A_1(t)) \otimes D(A_2(t))$, we have

$$\| A(t)u \|^2 = \|(A_1(t) \otimes I)(I \otimes A_2(t))u\|^2 \geqslant c_1{}^2 c_2{}^2 \| u \|^2,$$

and this inequality extends to $D(A(t))$, whence $A(t)$ satisfies (1.4). Furthermore, $\| A^{-1}(t) \| \leqslant (c_1 c_2)^{-1}$ for all $t \in [0, T]$.

If $\psi \in E$ and $\xi \in F$, then

$$\frac{d}{dt}(A^{-1}(t)(\psi \otimes \xi), \psi \otimes \xi)_H = (\dot{A}_1^{-1}(t)\psi, \psi)(A_2^{-1}(t)\xi, \xi) + (A_1^{-1}(t)\psi, \psi)(\dot{A}_2^{-1}(t)\xi, \xi).$$

Then using the bilinearity of the scalar product it follows that for $f, g \in E \otimes F$, we have

$$\frac{d}{dt}(A^{-1}(t)f, g) = ((\dot{A}_1^{-1}(t) \otimes A_2^{-1}(t))f, g) + ((A_1^{-1}(t) \otimes \dot{A}_2^{-1}(t)f, g).$$

Lemma 1.1 implies that there are constants c_1' and c_2' such that

$$\| \dot{A}_1^{-1}(t) \|_{\mathscr{L}(E,E)} \leqslant c_1' \qquad \text{and} \qquad \| \dot{A}_2^{-1}(t) \|_{\mathscr{L}(F,F)} \leqslant c_2'.$$

The operators $\dot{A}_1^{-1} \otimes A_2^{-1}(t)$ and $A_1^{-1}(t) \otimes \dot{A}_2^{-1}(t)$ both extend as continuous linear operators on H, denoted by $\dot{A}_1^{-1}(t) \hat{\otimes} A_2^{-1}(t)$ and $A_1^{-1}(t) \hat{\otimes} \dot{A}_2^{-1}(t)$. It follows, by a procedure similar to that used to show that $A_1(t) \otimes A_2(t)$ is one-to-one, that

$$\| \dot{A}_1^{-1}(t) \hat{\otimes} A_2^{-1}(t) \|_{\mathscr{L}(H,H)} \leqslant c_1' c_2^{-1}$$

and

$$\| A_1^{-1}(t) \hat{\otimes} \dot{A}_2^{-1}(t) \|_{\mathscr{L}(H,H)} \leqslant c_2' c_1^{-1} .$$

Thus

$$\left| \frac{d}{dt}(A^{-1}(t)f, g)_H \right| \leqslant (c_1' c_2^{-1} + c_2' c_1^{-1}) \| f \| \| g \|$$

for $f, g \in E \otimes F$. An application of the Banach–Steinhaus Theorem then shows that $(A^{-1}(t)f, g)_H$ is continuously differentiable on $[0, T]$ with the same estimate on the derivative. Thus (1.5) is satisfied.

Let $B_1(t)$ and $B_2(t)$ be the "absolute value" of $A_1(t)$ and $A_2(t)$, respectively. Using arguments similar to those already employed, it is readily verified that the operator $B(t) = B_1(t) \widehat{\otimes} B_2(t)$, the closure of $B_1(t) \otimes B_2(t)$, is the "absolute value" of $A(t)$. Since $B_1(t)$ and $B_2(t)$ satisfy (2.1) it follows that $B(t)$ also satisfies (2.1). This completes the proof of the theorem.

Before considering concrete examples we must provide a way to verify the hypothesis (2.1) in a practical situation. To this end we prove the following theorem.

Let $A(t)$ be a family of self-adjoint operators ($t \in [0, T]$) which satisfies (1.4) and (1.5). Let $B(t)$, as usual, be the "absolute value" of $A(t)$. $B(t)$ is, of course, a positive self-adjoint operator for $t \in [0, T]$, and thus the fractional powers $B^\rho(t)$, $0 \leqslant \rho \leqslant 1$, are defined.

(5.4) Suppose there exist constants $c > 0$ and $\rho > 0$ such that the range of $\dot{A}^{-1}(t)$ is contained in $D(B^\rho(t))$ for all $t \in [0, T]$, and that the resulting bounded operator $B^\rho(t)\, \dot{A}^{-1}(t)$ satisfies

$$\| B^\rho(t)\, \dot{A}^{-1}(t) \|_{\mathscr{L}(H,H)} \leqslant c \qquad \text{for all } t \in [0, T].$$

THEOREM 5.3. *Let $A(t)$ be a family of self-adjoint operators in a separable Hilbert space H, $t \in [0, T]$, satisfying* (1.4), (1.5), *and* (5.4). *Then $B(t)$ satisfies* (2.1).

Proof. We shall first obtain an expression for $B^{-1}(t)$ in terms of a contour integral.

It follows easily from (1.5) that $A^{-2}(t)$ is also weakly differentiable in the sense of (1.5) and that

$$\dot{A}^{-2}(t) = \dot{A}^{-1}(t)\, A^{-1}(t) + A^{-1}(t)\, \dot{A}^{-1}(t).$$

Then Lemma 1.1 implies that $t \to A^{-2}(t)$ is continuous in the uniform operator norm. Hence there is a constant $\delta > 0$ such that $\| A^{-2}(t) \| \leqslant \delta^{-1}$ for all $t \in [0, T]$, and the spectrum of $A^2(t)$ is contained in $\{x \in \mathbf{R} : x \geqslant \delta\}$. Now let γ be the following contour in the resolvant of $A^2(t)$, running from $-i\infty$ to $+i\infty$, and consisting of three pieces:

$$\gamma_1 = \{\lambda = iy : y \geqslant \delta/2\},$$

$$\gamma_2 = \{\lambda = x + iy : |\lambda| = \delta/2 \text{ and } x \geqslant 0\},$$

$$\gamma_3 = \{\lambda = iy : y \leqslant -\delta/2\}.$$

Then we may write

$$B^{-1}(t) = [A^2(t)]^{-1/2} = -\frac{1}{2\pi i}\int_\gamma \lambda^{-1/2}R_\lambda(t)\,d\lambda$$

where $R_\lambda(t) = (\lambda - A^2(t))^{-1}$ (see Kato, Ref. [13]). The integral is absolutely convergent, since for $\lambda \in \gamma_1 \cup \gamma_3$, we have $\| R_\lambda(t)\| \leqslant 1/|\lambda|$.

To show that $B^{-1}(t)$ is differentiable, we would like to differentiate under the integral sign. Thus we must calculate the derivative of $R_\lambda(t)$, and since γ lies entirely in the resolvant of $A^2(t)$ for all $t \in [0, T]$, we may use Lemma 5.1. It follows that

$$\begin{aligned}\dot{R}_\lambda(t) &= -A^2(t)\,R_\lambda(t)\,\dot{A}^{-2}(t)\,A^2(t)\,R_\lambda(t)\\ &= -\,A^2(t)\,R_\lambda(t)\,A^{-1}(t)\,\dot{A}^{-1}(t)\,A^2(t)\,R_\lambda(t)\\ &\quad -\,A^2(t)\,R_\lambda(t)\,\dot{A}^{-1}(t)\,A^{-1}(t)\,A^2(t)\,R_\lambda(t).\end{aligned} \tag{5.5}$$

Let us estimate the uniform norm of the first term, using (5.4):

$$\begin{aligned}&A^2(t)\,R_\lambda(t)\,A^{-1}(t)\,\dot{A}^{-1}(t)\,A^2(t)\,R_\lambda(t)\\ &\quad = A(t)\,R_\lambda(t)\,B^{-\rho}(t)\cdot B^{\rho}(t)\,\dot{A}^{-1}(t)\,A^2(t)\,R_\lambda(t).\end{aligned}$$

By the Gelfand Theorem, we have that for $\lambda \in \gamma_1 \cup \gamma_3$,

$$\| A^2(t)\,R_\lambda(t)\,B^{-\rho}(t)\|_{\mathscr{L}(H,H)} \leqslant |\lambda|^{-(1+\rho)/2} \qquad \text{for} \quad \rho > 0.$$

Hence, using (5.4),

$$\| A^2(t)\,R_\lambda(t)\,A^{-1}(t)\,\dot{A}^{-1}(t)\,A^2(t)\,R_\lambda(t)\| \leqslant c\,|\lambda|^{-(1+\rho)/2}$$

A similar estimate holds for the second term in (5.5). Hence for $\lambda \in \gamma_1 \cup \gamma_3$,

$$\| \dot{R}_\lambda(t)\|_{\mathscr{L}(H,H)} \leqslant 2c\,|\lambda|^{-(1+\rho)/2}$$

We may therefore use the Lebesgue convergence theorem to differentiate under the integral sign, thereby obtaining

$$\frac{d}{dt}(B^{-1}(t)f, g) = -\frac{1}{2\pi i}\int_\gamma \lambda^{-1/2}(\dot{R}_\lambda(t)f, g)d, \qquad f, g \in H.$$

Since $t \to (\dot{R}_\lambda(t)f, g)$ is continuous on $[0, T]$, it follows that $B(t)$ satisfies (1.5). This completes the proof of Theorem 5.3.

At present the author does not know if (1.5) implies (2.1) without the hypothesis (5.4). One can easily see that (5.4) is not a necessary condition of (2.1) by considering multiplication operators in $L^2(0, 1)$. One should also remark that the proof of Theorem 5.3 shows that $B^{-\rho}(t)$ is differentiable in the sense of (1.5) for any $\rho > 1$, without assuming (5.4).

Finally we note that Medeiros [17] has proved, under different hypotheses, that if $A(t) \geqslant 0$ is a family of self-adjoint operators with constant domain D and $t \to A(t)u$ ($u \in D$) is differentiable, then so is $t \to A^{1/2}(t)u$.

We shall apply Theorem 5.3 in the following situation. Let Ω be a bounded open subset of $\mathbf{R}^n$ whose boundary Γ is a $\mathscr{C}^\infty$ manifold of dimension $n-1$. Let $\mathscr{C}_0^\infty(\Omega)$ denote the space of all $\mathscr{C}^\infty$ functions on Ω of compact support. $H^s(\Omega)$, $s \in \mathbf{R}$, will denote the Sobolev space of order s (see Sobolev, Ref. [19]). For s a positive integer, we have that

$$\| u \|^2_{H^s(\Omega)} = \sum_{|\alpha| \leqslant s} \int_\Omega | D^\alpha u |^2 \, dx,$$

α is an n-tuple, $\alpha = (\alpha_1, ..., \alpha_n)$, where $\alpha_j \geqslant 0$ are integers, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$,

$$D^\alpha u = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

By $H_0^s(\Omega)$, $s \geqslant 0$, we denote the closure of $\mathscr{C}_0^\infty(\Omega)$ in $H^s(\Omega)$.

Let $A(t)$ be a self-adjoint operator in $L^2(\Omega)$ such that

(5.6) For all $t \in [0, T]$, $H_0^m(\Omega) \subset D(A(t))$, some fixed $m > 0$, and for $u \in H_0^m(\Omega)$, $| A(t)u |_{L^2} \leqslant c \, | u |_{H^m}$, $c > 0$.

Suppose that $A(t)$ satisfies (1.4) and satisfies a smoothness condition slightly stronger than (1.5), viz.,

(5.7) For some $s_0 > 0$, $t \to A^{-1}(t)f$, $f \in H$, is weakly differentiable in $H^{s_0}(\Omega)$, and the derivative $t \to \dot{A}^{-1}(t)f$ is continuous in $H^{s_0}(\Omega)$.

Clearly (5.6) implies (1.5).

THEOREM 5.4. *Let* $A(t)$ *be a family* ($t \in [0, T]$) *of self-adjoint operators in* $H = L^2(\Omega)$ *which satisfies* (1.4), (5.6), *and* (5.7). *Then* $A(t)$ *satisfies* (5.4), *and therefore* (2.1).

Proof. Let $\Lambda : H_0^m(\Omega) \to L^2(\Omega)$ be the standard positive self-adjoint operator such that for $u \in H_0^m(\Omega)$, $| \Lambda u | = \| u \|_{H^m}$. Let $B(t)$ be the "absolute value" of $A(t)$. Then $D(\Lambda) \subset D(B(t))$ for all $t \in [0, T]$. Now for θ, $0 < \theta < 1$, such that $m(1-\theta) < 1/2$,

$$D(\Lambda^{1-\theta}) = [H_0^m, H]_\theta = H_0^{m(1-\theta)} = H^{m(1-\theta)}$$

(see Lions and Magenes, Ref. [15]). Choose θ_0, $0 < \theta_0 < 1$, such that $m(1 - \theta_0) < \min(1/2, s_0)$. This implies that $D(\Lambda^{1-\theta_0}) \supset H^{s_0}(\Omega)$. It follows by (5.7) that for each $t \in [0, T]$, $\Lambda^{1-\theta_0}\dot{A}^{-1}(t)$ is a bounded operator, and that for each $f \in H$, $t \to \Lambda^{1-\theta_0}\dot{A}^{-1}(t)f$ is weakly continuous in H. Then by the Banach–Steinhaus Theorem we have, for $t \in [0, T]$,

$$| \Lambda^{1-\theta_0}\dot{A}^{-1}(t)f |_H \leqslant c_1 | f |_H , \qquad f \in H.$$

Now $D(\Lambda) \subset D(B(t))$, and for $u \in D(\Lambda)$, we have, by (5.6),

$$| B(t)u | = | A(t)u | \leqslant c | \Lambda u |, \qquad c > 0 \text{ a constant.}$$

By one of the Heinz inequalities (see Kato, Ref. [12]), we have that $D(\Lambda^{1-\theta_0}) \subset D(B^{1-\theta_0}(t))$ and that

$$\| B^{1-\theta_0}(t)\, \Lambda^{\theta_0-1} \| \leqslant c, \qquad c > 0 \text{ a constant.}$$

Thus for $f \in H$, $\dot{A}^{-1}(t)f \in D(B^{1-\theta_0}(t))$ and

$$\begin{aligned} | B^{1-\theta_0}(t)\, \dot{A}^{-1}(t)f | &\leqslant \| B^{1-\theta_0}(t)\, \Lambda^{\theta_0-1} \| \, \| \Lambda^{1-\theta_0}\dot{A}^{-1}(t) \| \, | f |_H \\ &\leqslant c_1 c_2 | f |_H , \end{aligned}$$

which proves the theorem.

For a first concrete example, we let $\Omega = [0, 1]$ and $H = L^2(0, 1)$. We define the operator

$$(A(t)u)(x) = i \frac{du}{dx}$$

with $D(A(t))$ the space of functions $u \in H^1(0, 1)$ such that

$$u(1) = \beta(t)\, u(0),$$

where β is a continuously differentiable complex-valued function on $[0, T]$ such that $| \beta(t) | = 1$ for all $t \in [0, T]$ and $\beta(t) \neq 1$. It is easily verified that $A(t)$ is self-adjoint in H and that $A^{-1}(t)$ is given by

$$A^{-1}(t)f = i \left[\frac{1}{1 - \beta} \int_0^1 f\, ds - \int_0^x f\, ds \right].$$

Clearly $A^{-1}(t)f$ is strongly differentiable in $H^s(0, 1)$ for any $s \geqslant 0$, and $H_0^1(0, 1) \subset D(A(t))$ for all $t \in [0, T]$. Thus $A(t)$ satisfies (5.6) and (5.7). It follows by Theorem 5.3 and 5.4 that $A(t)$ satisfies (1.4), (1.5), and (2.1).

Writing $\beta(t) = \exp(2\pi i \theta(t))$, the eigenfunctions of $A(t)$ are found to be

$$\chi_n(x, t) = \exp(-2\pi i(n + \theta(t))x),$$

with eigenvalues $\lambda_n = 2\pi(n + \theta(t))$ for $n = 0, \pm 1, \pm 2, \ldots$. If we assume that $0 < \theta(t) < 1$ for $t \in [0, T]$, then $H_+(t)$ is the subspace of $L^2(0, 1)$ spanned by $\chi_n(x, t)$ for $n = 0, 1, 2, \ldots$, $H_-(t)$ is the subspace spanned by $\chi_n(x, t)$ for $n = -1, -2, \ldots$, and $D(A_+^{1/2}(t))$ is the space of $u(x) \in L^2(0, 1)$ such that

$$(u, \chi_n(t)) = 0 \quad \text{for } n < 0 \qquad \text{and} \qquad \sum_0^\infty n\, |(u, \chi_n(t))|^2 < \infty,$$

and similarly for $D(A_-^{1/2}(t))$.

Finally, it is easily seen that if

$$\sup_{0 \leqslant t \leqslant T} \left| \frac{\beta'}{(1 - \beta)^2} \right|$$

is sufficiently small, $A(t)$ will also satisfy (1.6). In this case Theorems 1.2 and 2.1 yield the following result:

Let $f(t, x) \in L^2((0, T) \times (0, 1))$, $u_0(x) \in D(A_+^{1/2}(0))$, and $u_T(x) \in D(A(_-^{1/2}(T))$ be given. Then there is one and only one function $u(t, x)$ such that

(i) $u(t, x) \in L^2(0, T; H^1(0, 1)) \cap H^1(0, T; L^2(0, 1))$ and $u(1, x) = \beta(t)\, u(0, x)$ for almost all $t \in [0, T]$;

(ii) $\partial u/\partial t(t, x) + i(\partial u/\partial x)(t, x) = f(t, x)$ in $\mathscr{D}'((0, T) \times (0, 1))$;

(iii) $P_+(0)\, u(0, x) = u_0(x)$ and $P_-(T)\, u(T, x) = u_T(x)$.

As a second example, we again let $\Omega = [0, 1]$, $H = L^2(0, 1)$. We define the operator

$$(A(t)u)(x) = i\frac{d^3u}{dx^3}, \tag{5.8}$$

with $D(A(t))$ the space of functions $u(x) \in H^3(0, 1)$ such that

$$\begin{aligned} u_x(0) &= u_x(1), \\ u_{xx}(1) &= \beta(t)\, u(0), \\ u_{xx}(0) &= \beta(t)\, u(1), \end{aligned} \tag{5.9}$$

where $\beta(t)$ is a continuously differentiable function on $[0, T]$ such that $\beta(t) > 0$ for all $t \in [0, T]$. $A(t)$ is self-adjoint in H and $A^{-1}(t)$ is given by

$$A^{-1}(t)f = c_1(t) + c_2(t)x + c_3(t)\, x^2 + i\int_0^x \int_0^y \int_0^z f(\xi)\, d\xi\, dz\, dy$$

for $f \in H$, where

$$c_1(t) = \frac{1}{\beta(t)} [i \int_0^1 f(\xi)\, d\xi - i \int_0^1 \int_0^y f(\xi)\, d\xi\, dy],$$

$$c_2(t) = \frac{i}{2} \int_0^1 \int_0^y f(\xi)\, d\xi\, dy - i \int_0^1 \int_0^y \int_0^\xi f(s)\, ds\, d\xi\, dy - \frac{i}{\beta(t)} \int_0^1 f(\xi)\, d\xi,$$

$$c_3(t) = \frac{-i}{2} \int_0^1 \int_0^y f(\xi)\, d\xi\, dy.$$

For λ a real number we let α_1, α_2, α_3 denote the three cube roots of $-i\lambda$. Then the eigenfunctions of $A(t)$ are of the form

$$\omega(x) = b_1 \exp(\alpha_1 x) + b_2 \exp(\alpha_2 x) + b_3 \exp(\alpha_3 x)$$

where the $b_i = b_i(t)$ satisfy the equations

$$b_1\alpha_1(1 - \exp(\alpha_1)) + b_2\alpha_2(1 - \exp(\alpha_2)) + b_3\alpha_3(1 - \exp(\alpha_3)) = 0,$$

$$b_1({\alpha_1}^2 \exp(\alpha_1) - \beta(t)) + b_2({\alpha_2}^2 \exp(\alpha_2) - \beta(t)) + b_3({\alpha_3}^2 \exp(\alpha_3) - \beta(t)) = 0,$$

$$b_1({\alpha_1}^2 - \beta(t) \exp(\alpha_1)) + b_2({\alpha_2}^2 - \beta(t) \exp(\alpha_2)) + b_3({\alpha_3}^2 - \beta(t) \exp(\alpha_3)) = 0.$$

One can show that $(A(t)u, u)$ is not bounded, either above or below, using the functions $u_n = \cos n\pi x + i \sin n\pi x$, suitably modified to satisfy the boundary conditions (5.9). Thus there is an infinite sequence of values of λ, depending on t,

$$\cdots < \lambda_{-2} < \lambda_{-1} < 0 < \lambda_1 < \lambda_2 < \cdots$$

tending to $+\infty$ and $-\infty$, which make the equations for the $b_i(t)$ solvable. We label the corresponding eigenfunctions $\omega_n(t, x)$. Then $H_+(t)$ and $H_-(t)$ are the infinite-dimensional spaces spanned by $\omega_n(t, x)$ for $n \geqslant 0$ and $n < 0$, respectively.

$A^{-1}(t)f$ is strongly differentiable in $H^s(0, 1)$ for any $s \geqslant 0$ and ${H_0}^3(0, 1) \subset D(A(t))$ for all $t \in [0, T]$. Thus $A(t)$ defined by (5.8) and (5.9) satisfies (5.6) and (5.7). It follows as in the previous example that $A(t)$ satisfies (1.4), (1.5), and (2.1), and that if $\sup_{0 \leqslant t \leqslant T} | d/dt\, (1/\beta(t)|$ is sufficiently small, $A(t)$ will also satisfy (1.6). In this case Theorems 1.2 and 2.1 yield the result:

Let $f(t, x) \in L^2((0, T) \times (0, 1))$, $u_0(x) \in D(A_+^{1/2}(0))$, and $u_T(x) \in D(A_-^{1/2}(T))$ be given. Then there is one and only one function $u(t, x)$ such that

(i) $u(t, x) \in L^2(0, T; H^3(0, 1)) \cap H^1(0, T; L^2(0, 1))$ and $u(t, x)$ satisfies (5.9) in x for almost all $t \in [0, T]$;

(ii) $\partial u/\partial t\,(t, x) + i(\partial^3 u/\partial x^3)(t, x) = f(t, x)$ in $\mathscr{D}'((0, T) \times (0, 1))$;

(iii) $P_+(0)\, u(0, x) = u_0(x), P_-(T)\, u(T, x) = u_T(x)$.

Now using Theorems 5.1 and 5.2 we may construct more complicated examples. For instance, let $A_1(t) = A(t)$ defined by (5.8) and (5.9) in $E = L^2(0, 1)$. Let $\Omega = \{y \in \mathbf{R}^2 : |y| < 1\}$, $\Gamma = \{y \in \mathbf{R}^2 : |y| = 1\}$, and (with $y \in \mathbf{R}^2$ written $y = (y_1, y_2)$)

$$a(t, u, v) = \int_\Omega \left(\frac{\partial u}{\partial y_1}\frac{\partial \bar{v}}{\partial y_1} + \frac{\partial u}{\partial y_2}\frac{\partial \bar{v}}{\partial y_2}\right) dy_1\, dy_2 + \int_\Omega u\bar{v}\, dy_1\, dy_2$$

$$+ \gamma(t) \int_\Gamma u(s)\, \bar{v}(s)\, ds$$

be defined on $H^1(\Omega) \times H^1(\Omega)$, where $\gamma(t)$ is a real-valued continuously differentiable function on $[0, T]$ with $\gamma(t) > 0$ for all $t \in [0, T]$. Then

$$a(t, u, u) \geqslant \| u \|_H^2\, H^1(\Omega) \qquad \text{for all } t \in [0, T].$$

With $V(t) = K = H^1(\Omega)$ it is clear that the hypotheses (5.1), (5.2), and (5.3) are satisfied. Let $A_2(t)$ be the self-adjoint operator in $F = L^2(\Omega)$ defined by $a(t, u, v)$ and $H^1(\Omega)$. Then by Theorem 5.1, $A_2(t)$ satisfies (1.4), (1.5), and (2.1) because $A_2(t) \geqslant I > 0$. Now

$$A_2(t)u = -\Delta_y u + u$$

and

$$D(A_2(t)) = \left\{u \in H^2(\Omega) : \frac{\partial}{\partial \nu}\, u(s) = \gamma(t)\, u(s)\ \forall s \in \Gamma\right\},$$

where $\partial/\partial\nu$ is the exterior normal derivative. Thus by Theorem 5.2, the operators

$$(A_1(t) \mathbin{\hat{\otimes}} A_2(t))\, u(x, y) = -i \frac{\partial^3}{\partial x^3} \Delta_y u(x, y) + i \frac{\partial^3 u}{\partial x^3}(x, y)$$

with domain $D(A_1(t) \mathbin{\hat{\otimes}} A_2(t))$ also satisfy (1.4), (1.5), and (2.1).

In the following example we consider a family of operators which are indeed semibounded for each $t \in [0, T]$, but not uniformly. We are able to verify (2.1) by using Theorems 5.3 and 5.4. We again let $H = L^2(0, 1)$ and we define a family of operators $A(t)$ in H by

$$A(t)u = -u_{xx} - \tfrac{1}{4}u, \tag{5.10}$$

with $D(A(t))$ the space of functions $u \in H^2(0, 1)$ such that

$$(5.11) \qquad \begin{aligned} a(t)\, u(0) + b(t)\, u_x(0) &= 0, \\ a(t)\, u(1) + b(t)\, u_x(1) &= 0, \end{aligned}$$

where $a(t)$ and $b(t)$ are real-valued, continuously differentiable functions on $[0, T]$ such that

(i) $0 \leqslant a(t) \leqslant 1$ for $t \in (0, T)$; $a(0) = 0$, $a(T) = 1$;

(ii) $0 < b(t) \leqslant 1$ for $t \in (0, T)$; $b(0) = 1$, $b(T) = 0$.

$A(t)$ is self-adjoint, and its inverse may be calculated explicitly. For $f \in L^2(0, 1)$,

$$A^{-1}(t)f = C(t)\, e^{ix/2} + D(t)\, e^{-ix/2} - e^{-ix/2} \int_0^x \int_0^y e^{(iy - is/2)} f(s)\, ds\, dy$$

with

$$C(t) = \frac{-(a - \frac{1}{2}ib) \int_0^1 \int_0^y e^{(iy-is/2)} f(s)\, ds\, dy + b(t) \int_0^1 e^{-is/2} f(s)\, ds}{2(a(t) + \frac{1}{2}ib(t))},$$

$$D(t) = \frac{[a(t) - \frac{1}{2}ib(t)] \int_0^1 \int_0^y e^{i(y-s/2)} f(s)\, ds\, dy - b(t) \int_0^1 e^{-is/2} f(s)\, ds}{2(a(t) + \frac{1}{2}ib(t))}.$$

Here again $t \to A^{-1}(t)f$ is strongly differentiable in $H^s(0, 1)$ for all $s \geqslant 0$, and $H_0^2(0, 1) \subset D(A(t))$ for all $t \in [0, T]$. Hence by Theorems 5.3 and 5.4, $A(t)$ satisfies (1.4), (1.5), and (2.1). Easy estimates on $C'(t)$ and $D'(t)$ show that (1.6) is satisfied if

$$(5.12) \qquad \sup_{0 \leqslant t \leqslant T} \frac{|\, a'b - ab' \,|}{|\, a + \frac{1}{2}ib \,|^2} \leqslant 2\alpha, \qquad 0 \leqslant \alpha < 1.$$

At $t = 0$, $a(t) = 0$, so that the boundary conditions are of Neumann type while at $t = T$, $b(T) = 0$, yielding a Dirichlet condition. That this transition from the Neumann to the Dirichlet condition is of a singular nature may be seen from the coefficient a/b in the sesquilinear forms $a(t, u, v)$ which define $A(t)$ for $0 \leqslant t < T$,

$$(5.13) \quad a(t, u, v) = \int_0^1 u_x \bar{v}_x\, dx - \tfrac{1}{4} \int_0^1 u\bar{v}\, dx + \frac{a(t)}{b(t)} [u(1)\, \bar{v}(1) - u(0)\, \bar{v}(0)], \qquad u, v \in H^1(0, 1).$$

Now in fact, for $0 \leqslant t < T$, $A(t)$ has one negative eigenvalue, viz.,

$\lambda_0(t) = -\frac{1}{4} - (a/b)^2$, corresponding to the eigenfunction $e^{-ax/b}$. At $t = T$, $A(T) \geqslant \frac{3}{4}$, i.e., $H_-(T) = \{0\}$. Thus as $t \to T$, $\lambda_0(t) \to -\infty$, and the negative eigenvalues "disappear."

If $B(0)$ is the "absolute value" of $A(0)$, then it is easily seen that $D(B^{1/2}(0)) = [D(B(0)), L^2(0, 1)]_{1/2} = H^1(0, 1)$ (see Grisvard, Ref. [11]). The projection on the negative eigenspace, i.e., $P_-(0)$, is simply

$$P_-(0)f = \int_0^1 f(s)\, ds, \qquad f \in L^2(0, 1),$$

because the negative eigenfunction is $\varphi(x) = 1$. Hence

$$(P_+(0)f)(x) = f(x) - \int_0^1 f(s)\, ds.$$

It follows that $D(A_+^{1/2}(0)) = \{u \in H^1(0, 1) : \int_0^1 u\, ds = 0\}$.

Now suppose that (5.12) is valid and apply Theorem 2.3. Multiplying the solutions by $e^{-t/4}$, this yields the following result for the heat equation:

Let $f(t, x) \in L^2((0, T) \times (0, 1))$ and $u_0(x) \in H^1(0, 1)$ such that $\int_0^1 u_0(s)\, ds = 0$ be given. Then there is one and only one function $u(t, x)$ such that

(i) $u(t, x) \in L^2(0, T; H^2(0, 1)) \cap H^1(0, T; L^2(0, 1))$ and $u(t, x)$ satisfies (5.11) in x for almost all $t \in [0, T]$;

(ii) $\partial u/\partial t\, (t, x) - \partial^2 u/\partial x^2\, (t, x) = f(t, x)$ in $\mathscr{D}'((0, T) \times (0, x))$;

(iii) $u(0, x) - \int_0^1 u(0, s)\, ds = u_0(x)$.

Final Remarks. The question of determining when self-adjoint operators $A(t)$ satisfying (1.4) and (1.5) also satisfy (4.1) or (4.1)′ needed for the backward uniqueness theorem 4.1 is still open, except in the case of uniformly semibounded operators of Theorem 5.1. In the last example discussed, $A(t)$ defined by (5.10) and (5.11) arises from the forms (5.13) for $0 \leqslant t < T$. It can be shown that for each τ, $0 < \tau < T$, there is a $\lambda_T > 0$ such that for $u \in H^1(0, 1)$,

$$a(t, u, u) + \lambda_\tau \mid u \mid^2 \geqslant c \parallel u \parallel_{H^1}^2, \qquad 0 \leqslant t \leqslant \tau.$$

It follows by Theorem 5.1 that $A(t)$ satisfies (4.1)′ on each interval $[0, \tau]$, $0 < \tau < T$. A simple argument then shows that the solutions of $Au + u' = 0$, $u \in W$, where $A(t)$ is defined by (5.10) and (5.11), satisfy (4.3) on the whole interval $[0, T]$.

Another question remaining open is to determine the meaning of the restriction (1.6). It is not clear whether (1.6) is a hypothesis imposed by the method, or instrinsic to the problem. We note, however, that only (1.4), (1.5), and (2.1) are used in proving the trace result of Lemma 2.3. When $A(t) > 0$ for all $t \in [0, T]$, a more qualitative condition for existence and uniqueness in the forward Cauchy problem may be substituted for (1.6), viz., that $t \to A^{-1/2}(t)$ be weakly, continuously differentiable (see Carroll, Refs. [5, 7]). It would be desirable to find a comparable condition for the two-point problem.

Finally, we wish to mention the paper of Baouendi and Grisvard, Ref. [2] (cf. also Bardos and Brezis, Ref. [3]), which considers the equation

$$x \frac{\partial u}{dt}(x, t) - \frac{\partial^2 u}{\partial x^2} x, t) = f(x, t) \tag{5.14}$$

in the rectangle $[-1, 1] \times [0, T]$ with the boundary and initial conditions

$$\begin{aligned} u(1, t) &= u(-1, t) = 0 && \text{for} \quad 0 \leqslant t \leqslant T, \\ u(x, 0) &= u_0(x) && \text{for} \quad 0 \leqslant x \leqslant 1, \\ u(x, T) &= u_T(x) && \text{for} \ -1 \leqslant x \leqslant 0, \end{aligned} \tag{5.15}$$

where $u_0(x)$ and $u_T(x)$ are given functions. This is also a two-point problem, and the appropriate operator is $-1/x(d^2/dx^2)$. However, this operator is not even formally self-adjoint, and so does not fall within the scope of our theory.

On the other hand, one can construct an abstract framework which yields *weak* solutions to the two-point problem treated in this paper, as well as a class of problems including (5.14), (5.15). This is done by considering equations of the form

$$E(t) \frac{du}{dt} + B(t)\, u(t) = f(t),$$

where $B(t)$ is an accretive operator in H, and $E(t)$ is a symmetric bounded operator which decomposes H into "positive" and "negative" spaces $H_+(t)$ and $H_-(t)$. Details of this discussion will appear elsewhere.

Acknowledgments

I wish to thank Professor Robert Carroll, whose many suggestions were extremely helpful to the author during the preparation of an earlier version of this paper. I also wish to thank Professor J. L. Lions for helpful criticism of this article.

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